

MODULI INTERPRETATION OF EISENSTEIN SERIES

KAMAL KHURI-MAKDISI

ABSTRACT. Let $\ell \geq 3$. Using the moduli interpretation, we define certain elliptic modular forms of level $\Gamma(\ell)$, which make sense over any field k in which $6\ell \neq 0$ and that contains the ℓ th roots of unity. Over the complex numbers, these forms include all holomorphic Eisenstein series on $\Gamma(\ell)$ in all weights, in a natural way. The graded ring \mathcal{R}_ℓ that is generated by our special modular forms turns out to be generated by certain forms in weight 1 that, over \mathbf{C} , correspond to the Eisenstein series on $\Gamma(\ell)$. By a combination of algebraic and analytic techniques, including the action of Hecke operators and nonvanishing of L -functions, we show that when $k = \mathbf{C}$, the ring \mathcal{R}_ℓ , which is generated as a ring by the Eisenstein series of weight 1, contains all modular forms on $\Gamma(\ell)$ in weights ≥ 2 . Our results give a straightforward method to produce models for the modular curve $X(\ell)$ defined over the ℓ th cyclotomic field, using only exact arithmetic in the ℓ -torsion field of a single \mathbf{Q} -rational elliptic curve E_0 .

1. INTRODUCTION

Let L be a lattice in \mathbf{C} , and consider elliptic functions with respect to L . A standard formula (see, e.g., equation IV.3.6 of [Cha85]), which we reprove in Corollary 3.6 of this article, states that if $\alpha, \beta, \gamma \in \mathbf{C} - L$ and $\alpha + \beta + \gamma = 0$, then

$$(1.1) \quad \frac{-1}{2} \cdot \frac{\wp'(\alpha) - \wp'(\beta)}{\wp(\alpha) - \wp(\beta)} = \zeta(\alpha) + \zeta(\beta) + \zeta(\gamma).$$

Here \wp and ζ are the Weierstrass \wp and zeta functions with respect to L ; our notation for elliptic functions follows [Cha85], unless otherwise specified. Let us temporarily call the above expression $\lambda = \lambda_{\alpha, \beta, \gamma, L}$. In terms of the projective embedding of the elliptic curve $E = \mathbf{C}/L$ as a plane cubic using \wp and \wp' , this essentially says that λ is the slope of the line (in the affine part of the plane) joining the images of α and β . After a short calculation, we obtain that λ can be written as the absolutely convergent series

$$(1.2) \quad \lambda = \zeta(\alpha) + \zeta(\beta) + \zeta(\gamma) = \sum'_{\omega \in L} \left(\frac{1}{\omega + \alpha} + \frac{1}{\omega + \beta} + \frac{1}{\omega + \gamma} - \frac{3}{\omega} \right),$$

where the notation \sum' means that one omits the term $3/\omega$ from the summand when $\omega = 0$. Note that the individual sums such as $\sum_{\omega} 1/(\omega + \alpha)$ do not converge; however, if $\alpha, \beta, \gamma \in \frac{1}{\ell}L$ for some integer ℓ , then the sums can be regularized by Hecke's method to obtain Eisenstein series of weight 1 on the congruence subgroup $\Gamma(\ell)$. After overcoming some analytic hurdles, we indeed show in Section 2 below that λ is the value of a suitable Eisenstein series of weight 1. As for Eisenstein series in weights 2 and 3, these can be related to values of the \wp and \wp' functions, in other words to the affine coordinates of the torsion points of E corresponding

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to α , β , and γ . This means that the values of Eisenstein series of weights up to 3 can be computed from the Weierstrass model of the varying elliptic curve E and its ℓ -torsion, in other words from the moduli problem that is parametrized by the modular curve $X(\ell)$. This is the “moduli interpretation” referred to in our title.

More generally, we can search for other “moduli-friendly” modular forms on $\Gamma(\ell)$ that have an agreeable modular interpretation; this allows us to define such modular forms over more general base fields than \mathbf{C} . In truth, the first paragraph above reverses the order in which this author came across the family of moduli-friendly forms treated in this article. The realization that λ was a modular form came first, from different considerations. Indeed, the expression of λ as a slope quotient shows that it is the quotient of an Eisenstein series of weight 3 by one of weight 2; thus viewing λ as a function of the varying L (as well as α, β, γ), we obtain that λ is a meromorphic modular form of weight 1. Holomorphy of λ on the upper half plane \mathcal{H} and at the cusps then follows from the addition formula on the elliptic curve, namely, from the formula $\lambda^2 = \wp(\alpha) + \wp(\beta) + \wp(\gamma)$, which expresses λ^2 as a holomorphic modular form. It was in this way that we first collected a family of moduli-friendly forms (Definition 3.1, equation (3.23) and Theorem 3.9 below), defined more generally than in (1.1) by the coefficients in the Laurent expansions of certain elliptic functions, or rather the algebraic Laurent expansions of certain elements of the function field of E . It was only later in our investigations that we made the connection from these forms to the Weierstrass ζ function and Eisenstein series (Theorem 2.8, which now comes earlier in our treatment).

In particular, this article provides a moduli-friendly, algebraic treatment of all holomorphic Eisenstein series of arbitrary weight j on $\Gamma(\ell)$, and gives a natural way to express these Eisenstein series as polynomials in the Eisenstein series of weight 1. This occupies Sections 2 and 3 of this article. In fact, all the modular forms that we obtain belong to a ring \mathcal{R}_ℓ , which turns out to be generated by the algebraic version of the Eisenstein series in weight 1 for $\ell \geq 3$ (Theorem 3.13). This result is similar to the results proved in [BG01a], where Borisov and Gunnells define and study “toric modular forms” on $\Gamma_1(\ell)$, and prove that the ring of toric modular forms is generated by certain Eisenstein series in weight 1, and that it is stable under the Hecke operators T_n for $\Gamma_1(\ell)$; their proofs rely on q -expansions of modular forms. Thus the results in this article include a generalization to $\Gamma(\ell)$ of the ring of toric modular forms introduced in [BG01a]. (See also [Cor97] for a study of the ring generated by weight 1 Eisenstein series in the Drinfeld modular case.) The above article [BG01a], as well as the subsequent articles [BG01b, BG03, BGP01], were a definite inspiration for several of the results in this article, even though our proofs tend to proceed along different lines (most notably, without any q -expansions).

Sections 4 and 5 contain the main technical results of this article. Continuing the analogy with [BG01a], we also prove invariance of our ring \mathcal{R}_ℓ under the Hecke algebra. We first use an algebraic method to prove this in weights 2 and 3 (Propositions 4.6, 4.8, and 4.11, with whose proofs we are rather pleased). We then combine the Hecke invariance in these low weights with analytic techniques (Rankin-Selberg and nonvanishing of L -functions), along with some algebraic geometry of sufficiently positive line bundles on curves, to prove that over \mathbf{C} , the ring \mathcal{R}_ℓ contains *all* modular forms of weights $j \geq 2$ (Theorem 5.1). In weight 1, our ring contains precisely the Eisenstein series. Thus our ring is of necessity stable under Hecke operators, this time for the full (noncommutative) Hecke algebra of all double cosets for $\Gamma(\ell)$.

Our result that \mathcal{R}_ℓ contains all modular forms in higher weights is analogous to the results in [BG01b, BG03] for toric modular forms on $\Gamma_1(\ell)$. Borisov and Gunnells prove there that the cuspidal part of the toric modular forms in weight 2 consists of all cusp forms with nonvanishing central L -value, while in weight $j \geq 3$, the cuspidal part is all of $\mathcal{S}_j(\Gamma_1(\ell))$. Their approach also uses nonvanishing of L -functions, but is otherwise somewhat different; see the introduction to Section 5 below. In that section, we use our results so far to study models of the modular curve $X(\ell)$. We use our moduli-friendly interpretation of the elements of \mathcal{R}_ℓ to show the final result of this article (Theorem 5.5), which can be stated in the following striking manner: for $\ell \geq 3$, the slopes of lines joining the ℓ -torsion points of any *one* elliptic curve over \mathbf{Q} with $j \neq 0, 1728$ (for example, $E_0 : y^2 = x^3 + 3141x + 5926$) contain enough information to deduce equations for $X(\ell)$, which parametrizes the ℓ -torsion of *all* elliptic curves. Moreover, the computations involved to find the equations for $X(\ell)$ are all exact computations in the number field $\mathbf{Q}(E_0[\ell])$, and yield a model for $X(\ell)$ over the cyclotomic field $\mathbf{Q}(\mu_\ell)$. In particular, no infinite series or other approximations in \mathbf{C} are necessary.

Since our results are moduli-friendly and largely algebraic as opposed to analytic (except for Theorem 5.1), the approach in this article has the advantage that large parts of the theory work over more general fields k than \mathbf{C} , provided that 6ℓ is invertible in k , and that the ℓ th roots of unity are contained in k . Our approach also has the benefit of yielding a more direct connection to Eisenstein series and to moduli of elliptic curves without using q -expansions at any stage. We thus hope that the techniques we have developed can be of use in the study of modular forms over indefinite quaternion algebras and of Shimura curves.

To summarize, here are the main results in this article:

- A purely algebraic way to evaluate any Eisenstein series at a noncuspidal point $p \in X(\ell)$, in terms of a Weierstrass equation for the elliptic curve E_p corresponding to p in the moduli interpretation, along with the coordinates of the ℓ -torsion $E_p[\ell]$ (this is in Sections 2 and 3, which also include effectively computable expressions for Eisenstein series of any weight as polynomials in Eisenstein series of weight 1)
- An expression for Eisenstein series of weights 1 and 2 as absolutely convergent sums, without the need for Hecke's method of analytically continuing $\sum_{c,d} (c\tau + d)^{-j} |c\tau + d|^{-2s}$ in the parameter $s \in \mathbf{C}$ (Section 2)
- Several relations between the moduli-friendly forms, proved algebraically by a consideration of the moduli of elliptic curves (simpler relations in Section 3, and deeper relations in Section 4, which include the action of Hecke operators in weights 2 and 3)
- A proof that \mathcal{R}_ℓ contains all modular forms of weights ≥ 2 ; thus the only modular forms that are missed by \mathcal{R}_ℓ are the cusp forms of weight 1 (Theorem 5.1), which is in some sense not surprising, since these correspond to Galois representations of Artin type, and are the most intractable from an arithmetic viewpoint
- A systematic method to produce models for the curve $X(\ell)$ (Theorem 5.5; this model of a curve was called "Representation B" in [KM07], and allows for efficient computation in the Jacobian of $X(\ell)$). The idea is to use one fixed elliptic curve E_0 to produce sufficiently many points on a projective model for $X(\ell)$, so that only one curve $X(\ell)$ can reasonably interpolate

through all these points. This involves lengthy but purely algebraic computations in the field $\mathbf{Q}(E_0[\ell])$, and generalizes directly to all modular curves.

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2. EISENSTEIN SERIES AND LAURENT EXPANSIONS OF ELLIPTIC FUNCTIONS

Our first goal in this section is to describe a rearrangement of the sum in Eisenstein series that converges absolutely for all weights j , not just for $j \geq 3$. Let $\tau \in \mathcal{H}$, where \mathcal{H} is the complex upper half plane, and consider the lattice $L = L_\tau = \mathbf{Z} + \mathbf{Z}\tau$. Recall the definition of Eisenstein series on the principal congruence subgroup $\Gamma(\ell)$, with $\ell \geq 1$.

Definition 2.1. For $a_1, a_2 \in \mathbf{Z}$, let $\alpha = \alpha_\tau = (a_1\tau + a_2)/\ell \in \frac{1}{\ell}L_\tau$. For an integer $j \geq 1$ and for $s \in \mathbf{C}$, we define, following [Hec27], the Eisenstein series of weight j on $\Gamma(\ell)$:

$$\begin{aligned} (2.1) \quad G_j(\tau, \alpha; s) &= \sum_{\omega \in L_\tau} ' \frac{1}{(\alpha + \omega)^j |\alpha + \omega|^{2s}} \\ &= \sum_{(m,n) \in \mathbf{Z}^2} ' [(m + a_1/\ell)\tau + n + a_2/\ell]^{-j} |(m + a_1/\ell)\tau + n + a_2/\ell|^{-2s}, \end{aligned}$$

$$(2.2) \quad G_j(\tau, \alpha) = G_j(\tau, \alpha; 0), \quad \text{by analytic continuation.}$$

Here the notation \sum_{ω}' omits $\omega = -\alpha$ in case we have $\alpha \in L_\tau$; similarly for $\sum_{(m,n)}'$. The sum for $G_j(\tau, \alpha; s)$ converges for $\operatorname{Re} s > 1 - j/2$, and hence when $j \geq 3$ we have the absolutely convergent series $G_j(\tau, \alpha) = \sum_{\omega}' (\alpha + \omega)^{-j}$. For $j \geq 1$, Hecke showed that $G_j(\tau, \alpha; s)$ can be analytically continued to all $s \in \mathbf{C}$, and that $G_1(\tau, \alpha)$ is a holomorphic function of τ , while $G_2(\tau, \alpha)$ is the sum of $-2\pi i/(\tau - \bar{\tau})$ and a holomorphic function of τ . Since $G_j(\tau, \alpha; s)$ depends only on the class of α modulo L_τ , we can view α as an ℓ -torsion point on the elliptic curve $E = E_\tau = \mathbf{C}/L_\tau$. We shall nonetheless take care to distinguish between $\alpha \in \mathbf{C}$ and its image $P_\alpha \in E$.

We reformulate our Eisenstein series in terms of divisors on \mathbf{C} and on E . We establish the following notation to distinguish the notation for the formal sums of points appearing in divisors from sums in \mathbf{C} and from the group operation on E :

- A divisor on \mathbf{C} will be written $\tilde{D} = \sum_{\alpha} m_{\alpha}(\alpha)$, and its image in E is $D = \sum_{\alpha} m_{\alpha}(P_{\alpha})$. Note that the α need not be distinct modulo L , so some cancellation can occur in the formal sum for D . We call \tilde{D} a *lift* of D .
- The group operations of addition, inversion, and multiplication by an integer $n \in \mathbf{Z}$ on points $P, Q \in E$ are given by

$$(2.3) \quad P, Q \mapsto P \oplus Q, \quad P \mapsto \ominus P = [-1]P, \quad P \mapsto [n]P = P \oplus \cdots \oplus P, \text{ if } n \geq 1.$$

We denote by $P_0 \in E$ the additive identity in that group.

Definition 2.2. Let D be a divisor on E that is supported on the ℓ -torsion points $E[\ell]$, and choose any lift $\tilde{D} = \sum_{\alpha} m_{\alpha}(\alpha)$ of D to \mathbf{C} . We then define the following

Eisenstein series on $\Gamma(\ell)$:

$$(2.4) \quad G_j(\tau, D; s) = \sum_{\alpha} m_{\alpha} G_j(\tau, \alpha; s), \quad G_j(\tau, D) = G_j(\tau, D; 0).$$

It is immediate that the definition does not depend on the choice of lift \tilde{D} . We remind the reader that the values $\alpha \in \frac{1}{\ell}L_{\tau}$ (and corresponding points $P_{\alpha} \in E[\ell]$) vary with τ , as in Definition 2.1.

Our observation is that suitable choices of the lift \tilde{D} lead to series for $G_j(\tau, D; s)$ with good convergence for all $j \geq 1$. We motivate our discussion with the classical fact that a divisor $D = \sum_{\alpha} m_{\alpha}(P_{\alpha})$ on E is principal if and only if

$$(2.5) \quad \deg D := \sum_{\alpha} m_{\alpha} = 0, \quad \bigoplus D := \bigoplus_{\alpha} [m_{\alpha}]P_{\alpha} = P_0.$$

The latter sum above is evaluated in E .

Definition 2.3. Let D be a principal divisor on E . A *principal lift* of D is a divisor $\tilde{D} = \sum_{\alpha} m_{\alpha}(\alpha)$ on \mathbf{C} satisfying

$$(2.6) \quad \sum_{\alpha} m_{\alpha} = 0, \quad \sum_{\alpha} m_{\alpha} \alpha = 0 \quad (\text{both sums evaluated in } \mathbf{C}).$$

An arbitrary lift \tilde{D} would *a priori* merely satisfy $\sum_{\alpha} m_{\alpha} \alpha \in L$.

It is easy to see that principal lifts always exist. For example, if $\alpha = (a_1\tau + a_2)/\ell$, then the divisor $D = \ell(P_{\alpha}) - \ell(P_0)$ is principal, and all of the following are principal lifts of D :

$$(2.7) \quad \begin{aligned} \tilde{D}_1 &= \ell(\alpha) - (\ell - 1)(0) - (a_1\tau + a_2), \\ \tilde{D}_2 &= (\ell - 1)(\alpha) + (\alpha - a_1\tau - a_2) - \ell(0), \\ \tilde{D}_3 &= (\ell + 1)(\alpha) - (\alpha + a_1\tau + a_2) - \ell(0). \end{aligned}$$

Proposition 2.4. *Given a principal divisor D supported on $E[\ell]$, choose a principal lift \tilde{D} satisfying (2.6). Then*

$$(2.8) \quad \sum_{\alpha} \frac{m_{\alpha}}{(\alpha + \omega)^j |\alpha + \omega|^{2s}} = O\left(\frac{1}{|\omega|^{2s+j+2}}\right), \text{ for large } |\omega|.$$

We hence obtain for all $j \geq 1$ the following convergent double series (where the notation \sum'_{α} means that we omit $\alpha = -\omega$ if it appears in the inner sum):

$$(2.9) \quad G_j(\tau, D) = \sum_{\omega \in L} \sum'_{\alpha} \frac{m_{\alpha}}{(\alpha + \omega)^j |\alpha + \omega|^{2s}} \Big|_{s=0} = \sum_{\omega \in L} \left(\sum'_{\alpha} \frac{m_{\alpha}}{(\alpha + \omega)^j} \right).$$

Note that the outer sum over ω is absolutely convergent for $\operatorname{Re} s > -j/2$, even though the double sum converges only conditionally.

Proof. Define the \mathcal{C}^{∞} function $F(u) = \frac{1}{(u+\omega)^j |u+\omega|^{2s}} = (u+\omega)^{-j-s}(\bar{u}+\bar{\omega})^{-s}$, upon taking suitable branches for the powers. By Taylor's formula with respect to u and

\bar{u} , we have

$$(2.10) \quad F(\alpha) = F(0) + \frac{\partial F}{\partial u} \Big|_{u=0} \alpha + \frac{\partial F}{\partial \bar{u}} \Big|_{u=0} \bar{\alpha} + \int_{u=0}^{\alpha} \left[(\alpha - u) \frac{\partial^2 F}{\partial u^2} du + (\bar{\alpha} - \bar{u}) \frac{\partial^2 F}{\partial u \partial \bar{u}} du + (\alpha - u) \frac{\partial^2 F}{\partial u \partial \bar{u}} d\bar{u} + (\bar{\alpha} - \bar{u}) \frac{\partial^2 F}{\partial \bar{u}^2} d\bar{u} \right].$$

The integral is along any suitable path in the complex plane from 0 to α , say for example a line segment. We hence obtain an expansion valid for $|\omega| > 2|\alpha|$:

$$(2.11) \quad \frac{1}{(\alpha + \omega)^j |\alpha + \omega|^{2s}} = \frac{1}{\omega^j |\omega|^{2s}} - \frac{(s+j)\alpha}{\omega^{j+1} |\omega|^{2s}} - \frac{s\bar{\alpha}}{\omega^{j-1} |\omega|^{2s+2}} + O\left(\frac{1}{|\omega|^{2s+j+2}}\right).$$

Here the implied constant depends on α , j , and s , and is uniform in τ when τ is restricted to a compact subset of \mathcal{H} . Now multiply (2.11) by m_α , and sum over α to obtain (2.8) and (2.9). (Note that the sum \sum'_α does not omit any α once $|\omega|$ is sufficiently large). \square

Remark 2.5. Note that we always obtain holomorphic functions of τ above. In the setting of weight $j = 2$, this arises because we have always taken $\deg D = 0$, so the nonholomorphic terms cancel.

Proposition 2.4 allows us to rederive Hecke's second definition of weight 1 Eisenstein series as “division values” of the Weierstrass ζ function in Section 6 of [Hec26], as well as Corollary 3.4.24 of [Kat76]; we reprove those results in (2.14) below. Recall the absolutely and uniformly convergent series for $\zeta(z)$ for z in a compact subset of $\mathbf{C} - L_\tau$:

$$(2.12) \quad \zeta(z) = \frac{1}{z} + \sum_{0 \neq \omega \in L} \left[\frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right] = \frac{1}{z} + \sum_{0 \neq \omega \in L} \frac{z^2}{(z - \omega)\omega^2}.$$

It is a standard fact that $\zeta(z + m\tau + n) = \zeta(z) + 2m\eta_2 + 2n\eta_1$ for $m, n \in \mathbf{Z}$ (with “constants” $\eta_i = \eta_i(L)$ satisfying $2\eta_1\tau - 2\eta_2 = 2\pi i$). Here we follow the notation of Chapter IV of [Cha85]; note that Hecke and other authors use η_i for what we have called $2\eta_i$. Moreover, ζ is an odd function of z , and in fact its Laurent expansion near 0 is $\zeta(z) = z^{-1} + O(z^3)$.

Corollary 2.6. *Let D be a principal divisor supported on $E[\ell]$, and take a principal lift $\tilde{D} = \sum_\alpha m_\alpha(\alpha)$ for which every instance of P_0 in D is lifted to $\alpha = 0$. Then*

$$(2.13) \quad G_1(\tau, D) = \sum_{\alpha \neq 0} m_\alpha \zeta(\alpha).$$

Moreover, let $P_\alpha \in E[\ell] - \{P_0\}$, with any choice of lift $\alpha = (a_1\tau + a_2)/\ell$ with $a_1, a_2 \in \mathbf{Z}$. Then

$$(2.14) \quad \begin{aligned} G_1(\tau, P_\alpha) &= \zeta(\alpha) + \frac{1}{\ell} [\zeta(\alpha) - \zeta(\alpha + a_1\tau + a_2)] \\ &= \zeta\left(\frac{a_1\tau + a_2}{\ell}\right) - \frac{a_1}{\ell} \cdot 2\eta_2 - \frac{a_2}{\ell} \cdot 2\eta_1. \end{aligned}$$

Proof. Write $\tilde{D} = m_0(0) + \sum_{\alpha \neq 0} m_\alpha(\alpha)$, with $\alpha \neq 0 \implies \alpha \notin L$ by our assumption on \tilde{D} . Changing the sign of ω in (2.12), we obtain

$$(2.15) \quad \sum_{\alpha \neq 0} m_\alpha \zeta(\alpha) = \sum_{\alpha \neq 0} \frac{m_\alpha}{\alpha} + \sum_{\omega \neq 0} \sum_{\alpha \neq 0} \left[\frac{m_\alpha}{\alpha + \omega} - \frac{m_\alpha}{\omega} + \frac{m_\alpha \alpha}{\omega^2} \right].$$

The change of order of summation is justified by the good convergence of the series for ζ and because the sum over α is finite. Since \tilde{D} satisfies (2.6), we have $\sum_{\alpha \neq 0} m_\alpha = -m_0$ and $\sum_{\alpha \neq 0} m_\alpha \alpha = 0$, which allows us to rewrite the above sum in the form of (2.9) (at the cost of replacing absolute convergence with conditional convergence), and hence to obtain (2.13). Now apply this result in the case $D = \ell(\alpha) - \ell(0)$, using the principal lift \tilde{D}_3 from (2.7). This yields (2.14), because $G_1(\tau, \ell(\alpha) - \ell(0)) = \ell G_1(\tau, \alpha) - \ell G_1(\tau, 0)$ and $G_1(\tau, 0) = 0$ (more generally, $G_j(\tau, -\beta; s) = (-1)^j G_j(\tau, \beta; s)$). \square

We now turn to the second goal of this section, which is to relate Eisenstein series on $\Gamma(\ell)$ to Laurent expansions of elliptic functions.

Definition 2.7. Let D be a principal divisor on E , and let m_0 be the multiplicity of P_0 in D . We define an element f_D of the function field of E , which we also view as an elliptic function on \mathbf{C} with respect to L , by the requirements

$$(2.16) \quad \operatorname{div}(f_D) = D, \quad f_D = z^{m_0}(1 + O(z)), \quad \text{near } z = 0.$$

Here the first requirement determines f_D up to a nonzero constant factor, and the second requirement normalizes the constant so as to fix our choice of f_D . Our normalization ensures that for principal divisors D and E ,

$$(2.17) \quad f_{D+E} = f_D \cdot f_E.$$

We remark that the precise normalization of the constant factor in f_D will be needed in later sections of this article; it is not significant in this section, since we will mainly consider the logarithmic differential df_D/f_D .

Theorem 2.8. *Let D be a principal divisor, and take a principal lift $\tilde{D} = \sum_{\alpha} m_\alpha(\alpha)$. Make the same assumption on \tilde{D} as in Corollary 2.6. Then*

$$(2.18) \quad \frac{df_D}{f_D} = \sum_{\alpha} m_\alpha \zeta(z - \alpha) dz = \sum_{\omega \in L} \left[\sum_{\alpha} \frac{m_\alpha}{z - \alpha - \omega} \right] dz,$$

where the last series has similar convergence properties to the series of (2.9). Furthermore, if D is supported on $E[\ell]$, then the Laurent series expansion of df_D/f_D near $z = 0$ is

$$(2.19) \quad \frac{df_D}{f_D} = \left(\frac{m_0}{z} - \sum_{j \geq 1} G_j(\tau, D) z^{j-1} \right) dz.$$

Proof. It is classical (see, for example, Section IV.3 of [Cha85]) that we can express f_D up to a nonzero constant $C = C_\tau$ in terms of the Weierstrass σ function, provided that we have taken a principal lift \tilde{D} :

$$(2.20) \quad f_D(z) = C \prod_{\alpha} [\sigma(z - \alpha)^{m_\alpha}].$$

Taking logarithmic differentials yields the first equality in (2.18), since $\sigma'/\sigma = \zeta$. The second equality now follows from substituting the series for ζ and using the fact that $\sum_{\alpha} [m_{\alpha}/\omega + m_{\alpha}(z - \alpha)/\omega^2] = 0$.

We can now prove (2.19). The first term in the Laurent expansion is easy, and the other terms are equivalent to showing that $\text{Res}_{z=0} \left[z^{-j} \frac{df_D}{f_D} \right] = -G_j(\tau, D)$ for $j \geq 1$. This residue can be computed by a contour integral on a small circle enclosing $z = 0$. Since the sum over ω in (2.18) converges well, we are justified in computing the residue term-by-term, using the expansion $\frac{1}{z-\beta} = -\frac{1}{\beta} - \frac{z}{\beta^2} - \frac{z^2}{\beta^3} - \dots$ for $\beta \neq 0$ to compute residues for each inner sum over α that occurs as a term in the sum over ω . Comparing with (2.9) yields the desired result. \square

Remark 2.9. Note that (2.18) nicely confirms the fact (a simple consequence of (2.16)) that the differential form df_D/f_D is periodic with respect to L , has only simple poles, and has residue m_{α} at all points $\alpha + \omega$. We would have liked to use this fact to give a different proof of (2.19), by taking the contour integral of $z^{-j} \frac{df_D}{f_D}$ around a large circle with center at 0 and radius R (or perhaps using a large parallelogram). At least for $j \geq 2$, this approach works, since the contour integral tends to zero as $R \rightarrow \infty$. This explains the minus sign in our results, as well as the summation \sum' for Eisenstein series, which gives special treatment to the pole at $z = 0$ due to the presence of z^{-j} . However, we were not able to push through this argument for the important case $j = 1$. This is because an argument based only on the locations and residues of the poles of the differential form df_D/f_D cannot distinguish it from any other differential form $df_D/f_D + C dz$ where C is a constant.

The above theorem appears to relate Laurent expansions of elliptic functions only to those Eisenstein series $G_j(\tau, D)$ where D is principal. On the other hand, $G_j(\tau, D)$ depends \mathbf{Z} -linearly on D (in fact, so does df_D/f_D , by (2.17)), so we are led to consider linear combinations of Eisenstein series.

Proposition 2.10. *Let $\ell \geq 2$. Then for all $j \geq 1$, the \mathbf{C} -span of the Eisenstein series $\{G_j(\tau, D) \mid D \text{ principal, supported on } E[\ell]\}$ consists of all holomorphic Eisenstein series of weight j on $\Gamma(\ell)$.*

Proof. For all $P \in E[\ell]$, the divisor $\ell(P) - \ell(P_0)$ is principal, so the \mathbf{C} -span of our Eisenstein series contains all Eisenstein series of the form $G_j(\tau, P) - G_j(\tau, P_0)$. (If j is odd, then $G_j(\tau, P_0) = 0$ as we have already noted in the proof of Corollary 2.6, so we are done. But we will not use this fact). We conclude that our \mathbf{C} -span contains all combinations $\sum_{P \in E[\ell]} c_P G_j(\tau, P)$ for which $\sum_P c_P = 0$. If $j = 2$, then this is the space of all holomorphic Eisenstein series, since we want the nonholomorphic terms $2\pi i/(\tau - \bar{\tau})$ in G_2 to cancel. If $j \neq 2$, then it suffices to show that we can obtain $G_j(\tau, P_0)$ (which is of course an Eisenstein series on $\Gamma(1)$). To this end, consider the principal divisor $D = [\sum_{P \in E[\ell]} (P)] - \ell^2(P_0)$. We obtain $G_j(\tau, D) = (\ell^j - \ell^2)G_j(\tau, P_0)$, so we are done, since $\ell^j - \ell^2 \neq 0$ by our assumptions on ℓ and j . \square

Remark 2.11. The insistence on restricting to the case D principal is in fact a red herring, for deeper reasons than the above proposition. Take a more general D , which we assume for convenience is supported on $E[\ell]$ (although we can often manage with the weaker assumption that $\bigoplus D \in E[\ell]$, in the notation of (2.5)).

We can canonically replace D with $D - (\deg D)(P_0)$, which does not change $\bigoplus D$ but now gives us a divisor of degree zero.

We thus assume in this discussion that $\deg D = 0$, but that $\bigoplus D \in E[\ell]$ need not be trivial. Now the divisor ℓD is principal, and we can formally define $f_D = (f_{\ell D})^{1/\ell}$ for compatibility with (2.17). Note that if $\bigoplus D \neq P_0$, then f_D cannot be an elliptic function with respect to L ; its formal logarithmic derivative is nonetheless always periodic with respect to L , and we can simply take $df_D/f_D = (1/\ell)df_{\ell D}/f_{\ell D}$ as a definition. With this convention, (2.19) continues to hold, and we can obtain an analog of (2.18) as a series with good convergence properties, similarly to our derivation of (2.14).

We can however be more ambitious. Since $f_{\ell D}$ has zeros and poles with multiplicity everywhere divisible by ℓ , we see that f_D makes sense as a meromorphic function on \mathbf{C} . We use this to normalize the choice of ℓ th root f_D so that its Laurent series begins with z^{m_0} , as in (2.16). With our above conventions (especially in light of (2.17)), the f_D that we consider are products of (positive and negative) powers of the $f_P = (f_{\ell(P) - \ell(P_0)})^{1/\ell}$, for $P \in E[\ell] - \{P_0\}$. For such a “basic” f_P , Theorem 2.8 then states that

$$(2.21) \quad \begin{aligned} \frac{df_P}{f_P} &= z^{-1} \left(-1 - \sum_{j \geq 1} (G_j(\tau, P) - G_j(\tau, P_0)) z^j \right) dz \\ &= z^{-1} \left(-1 - G_1(\tau, P)z + (-G_2(\tau, P) + G_2(\tau, P_0))z^2 + \cdots \right) dz. \end{aligned}$$

Note that if $P = P_0$, then $f_{P_0} = 1$, so (2.21) no longer holds (indeed, $df_{P_0}/f_{P_0} = 0$, so the first coefficient in the Laurent expansion is now 0 instead of -1).

Even when D is nonprincipal as above, one can show that f_D is still an elliptic function, however with respect to the sublattice ℓL of L . When $D = P$, the behavior of f_P under translations by L is described by a Weil pairing; see Definition 4.1 in Section 4 below, where we work instead with the function $g_P(z) = f_P(\ell z)$, which is elliptic with respect to the original lattice L . One can similarly analyze the behavior of an arbitrary f_D under translations by L in terms of suitable Weil pairings. The approach of working with f_P that are periodic with respect to ℓL is used in the work of Borisov and Gunnells on toric modular forms [BG01a]. They use the function $\vartheta = \vartheta_{11}$ to write down what amounts to the same function as f_P when $P = a/\ell + L$ is in the subgroup of $E[\ell]$ generated by $P_{1/\ell}$. They then use the expansion of df_P/f_P at $z = 0$ to define their toric modular forms $s_{a/\ell}^{(k)}$ (see Section 4.4 of [BG01a]). Thus their $s_{a/\ell}^{(k)}$ are a special case of our $G_j(\tau, D)$, where the divisor D is of the form $[a]P_{1/\ell} - P_0$. This means that the $s_{a/\ell}^{(k)}$ are Eisenstein series with respect to $\Gamma_1(\ell)$ instead of $\Gamma(\ell)$; Borisov and Gunnells recognize this from the q -expansions, while our approach is more direct. Another advantage of our generalization to $\Gamma(\ell)$ is that for $\ell \geq 2$, we obtain the full space of holomorphic Eisenstein series of level $\Gamma(\ell)$, in all weights, by Proposition 2.10; see also Theorems 3.9 and 3.13 below. In contrast, the ring of toric modular forms on $\Gamma_1(\ell)$ does not always contain all Eisenstein series on that group: see Remark 4.13 of [BG01b].

Remark 2.12. One can find the Laurent expansion of f_D by formally exponentiating the integral of df_D/f_D . Keeping track of the algebra, one obtains that f_D has an expansion of the following form near $z = 0$:

$$(2.22) \quad f_D = z^{m_0} (1 + F_1(\tau)z + F_2(\tau)z^2 + \cdots),$$

where F_j is a modular form on $\Gamma(\ell)$ of weight j , expressible as a polynomial in the $G_j(\tau, D)$. This approach is used extensively in [BG01a]. In the next section, we study the Laurent series of f_D directly in a purely algebraic setting over a more general field k , and reformulate and extend the results of this section algebraically. For now, we simply note the result for f_P , obtained from (2.21):

$$(2.23) \quad f_P = z^{-1} \left[1 - G_1 z + \frac{(G_1^2 - \tilde{G}_2)}{2} z^2 - \left(\frac{G_3}{3} - \frac{G_1 \tilde{G}_2}{2} + \frac{G_1^3}{6} \right) z^3 + \dots \right]$$

where we wrote $G_1 = G_1(\tau, P)$, $\tilde{G}_2 = G_2(\tau, P) - G_2(\tau, P_0)$, and $G_3 = G_3(\tau, P)$ to save space. For the “genuine” elliptic function $f_{\ell(P) - \ell(P_0)} = f_P^\ell$, we have the expansion

$$(2.24) \quad f_{\ell(P) - \ell(P_0)} = z^{-\ell} (1 - \ell G_1(\tau, P) z + \dots).$$

Analogous results to (2.23) and (2.24) hold for arbitrary D .

3. ALGEBRAIC REFORMULATION AND THE RING \mathcal{R}_ℓ OF MODULAR FORMS

Our first step in “algebrizing” the results of the previous section is to normalize the equation of our elliptic curve E . We embed E into the projective plane \mathbf{P}^2 as follows (note the factor $1/2$):

$$(3.1) \quad z \mapsto P_z = [\wp(z; L) : (1/2)\wp'(z; L) : 1] = [x(z) : y(z) : 1].$$

As usual, $P_0 = [0 : 1 : 0]$ is the identity element. The affine algebraic equation of E and the invariant differential ω on E are

$$(3.2) \quad E : y^2 = x^3 + ax + b, \quad \omega = dx/(2y) = dz.$$

Here $a = a(\tau)$ and $b = b(\tau)$ are, up to constant factors, the Eisenstein series of level 1 and weights 4 and 6, respectively:

$$(3.3) \quad a(\tau) = -15G_4(\tau, 0) = -15 \sum_{0 \neq \omega \in L_\tau} \omega^{-4}, \quad b(\tau) = -35G_6(\tau, 0).$$

The symbol ω in (3.3) denotes an element of L , but for the rest of this article it will refer almost exclusively to the invariant differential, as in (3.2).

We now regard the family $\{E_\tau \mid \tau \in \mathcal{H}\}$ as a single elliptic curve E over the rational function field $\mathbf{C}(a, b)$ in two independent transcendental variables. We can work with more general fields k instead of \mathbf{C} ; in that case, E is a curve over the field $K = k(a, b)$. It is convenient to define the following graded rings, where a and b have weights 4 and 6, respectively:

$$(3.4) \quad \mathcal{R}_1 = k[a, b], \quad \mathcal{R}_{1, \mathbf{Z}} = \text{the image of } \mathbf{Z}[a, b] \text{ inside } \mathcal{R}_1.$$

Here \mathcal{R}_1 is of course an algebraic analog of the graded ring $\mathbf{C}[a(\tau), b(\tau)]$ of modular forms on the full modular group $\Gamma(1)$. Since we wish to use Weierstrass normal form for E , and also need to consider the ℓ -torsion throughout, we require 6ℓ to be invertible in k , and for k to contain the group μ_ℓ of ℓ th roots of unity (so as to accommodate the Weil pairing later). We extend scalars so that E is now defined over the ℓ -torsion extension field K_ℓ , a subfield of the algebraic closure \overline{K} of K :

$$(3.5) \quad K_\ell = K(E[\ell]) = K(\{x_P, y_P \mid P = (x_P, y_P) \in E[\ell](\overline{K}) - \{P_0\}\}).$$

Over \mathbf{C} , it is classical (Section 2 of [Hec27], especially equations (12–14)) that x_P and y_P are Eisenstein series of weights 2 and 3, respectively, when viewed as

functions of τ . Specifically, let $P = P_\alpha$ for $\alpha = \alpha_\tau \in \frac{1}{\ell}L_\tau - L_\tau$. Then the usual series for \wp and \wp' , along with (2.9), immediately give us

$$(3.6) \quad x_P = \wp(\alpha; L_\tau) = G_2(\tau, \alpha) - G_2(\tau, 0), \quad y_P = (1/2)\wp'(\alpha; L_\tau) = -G_3(\tau, \alpha).$$

We now turn to the algebraic Laurent expansions of meromorphic functions on E (i.e., of elements of the function field $K_\ell(E)$), but we also view these as elliptic functions with respect to L_τ when $k = \mathbf{C}$). We fix an algebraic uniformizer t at P_0 :

$$(3.7) \quad t = -x/y \quad (= z - 2az^5/5 + O(z^7) \text{ when } k = \mathbf{C}).$$

We also write $\hat{\mathcal{O}}$ for the completion of the local ring of E at P_0 ; hence $\hat{\mathcal{O}}$ is canonically isomorphic to the power series ring $K_\ell[[t]]$ — we occasionally tacitly extend scalars to work in $\overline{K}[[t]]$ — and we can view $\mathcal{R}_{1,\mathbf{Z}}[[t]]$ as a subring of $\hat{\mathcal{O}}$. (When k has characteristic zero, we can still make sense of the analytic uniformizer z as an element of $\hat{\mathcal{O}}$, since the relation $\omega = dz$ means that $z = \int \omega = t + 2at^5/5 + \dots$, from (3.8) below.) The meromorphic functions $x, y \in K_\ell(E[\ell])$ then have the following algebraic Laurent expansions:

$$(3.8) \quad \begin{aligned} x &= t^{-2} - at^2 + \dots = t^{-2}(1 - at^4 + \dots) \in t^{-2}\mathcal{R}_{1,\mathbf{Z}}[[t]], \\ -tx &= y = -t^{-3} + at + \dots = t^{-3}(-1 + at^4 + \dots) \in t^{-3}\mathcal{R}_{1,\mathbf{Z}}[[t]], \\ \omega &= (1 + 2at^4 + \dots)dt \in \mathcal{R}_{1,\mathbf{Z}}[[t]]dt. \end{aligned}$$

Moreover, the coefficient of t^j in the power series inside each pair of parentheses above is always a weight j homogeneous element of the graded ring $\mathcal{R}_{1,\mathbf{Z}}$. For all this, see for example Section IV.1 in [Sil86], as well as Lemma 3.8 below; alternatively, one can proceed starting from the usual analytic expansion of \wp in case $k = \mathbf{C}$ to obtain expansions of x, y , and t in terms of z , and then revert the series $t(z)$ to obtain series for z, x , and y in terms of t .

Our goal is now to study the algebraic Laurent expansions of the meromorphic functions $f_D \in K(E)$ of Definition 2.7. The second requirement in (2.16), normalizing the constant factor in f_D , now becomes $f_D = t^{m_0}(1 + O(t)) \in t^{m_0}(1 + t\hat{\mathcal{O}})$. This is compatible with our previous normalization when $k = \mathbf{C}$, since $t = z + O(z^2)$ by (3.7).

Definition 3.1. Let D be a principal divisor supported on $E[\ell]$, with m_0 the multiplicity of P_0 in D as before. For $j \geq 1$, we define $\lambda_D^{(j)}$ to be the following coefficient in the Laurent expansion of f_D at P_0 :

$$(3.9) \quad f_D = t^{m_0}(1 + \lambda_D^{(1)}t + \lambda_D^{(2)}t^2 + \dots) = t^{m_0}(1 + \lambda_D t + \mu_D t^2 + \nu_D t^3 + \dots).$$

In the above equation, we have also introduced the useful abbreviations

$$(3.10) \quad \lambda_D = \lambda_D^{(1)}, \quad \mu_D = \lambda_D^{(2)}, \quad \nu_D = \lambda_D^{(3)}.$$

We extend the above definitions to arbitrary D supported on $E[\ell]$ by the method of Remark 2.11: we form the degree zero divisor $D - (\deg D)(P_0)$, and multiply it by ℓ to obtain a principal divisor $D' = \ell \cdot [D - (\deg D)(P_0)]$. We then define

$$(3.11) \quad f_D = (f_{D'})^{1/\ell} = t^{m_0 - \deg D}(1 + \lambda_D t + \dots) \in t^{m_0 - \deg D}\hat{\mathcal{O}},$$

using the formal ℓ th root of the power series, and use this expansion to define the $\lambda_D^{(j)}$ in general. For D principal, this yields the same definition as before,

because (2.17) still holds. We can further use (2.17) to deduce various relations among the $\{\lambda_D^{(j)}\}$, most notably

$$(3.12) \quad \lambda_{D+E} = \lambda_D + \lambda_E.$$

By the discussion in Remark 2.12, each $\lambda_D^{(j)}$ is a modular form of weight j on $\Gamma(\ell)$ when $k = \mathbf{C}$; the fact that the expansions in (2.22)–(2.24) are with respect to z instead of t does not affect this statement. We nonetheless prefer to give an independent self-contained algebraic formulation and proof of this result. It is sufficient for this article to work with the following naive algebraic definition of modular forms; in contrast to the standard definition in, e.g., Section 2 of [Kat76], we evaluate our modular forms only on the pair (E, ω) with ℓ -torsion over the base field K_ℓ .

Definition 3.2. An algebraic modular form of level $\Gamma(\ell)$ and weight $j \geq 0$ is an element $f \in K_\ell$ satisfying the two properties:

- (1) We can write $f = g(\{x_P, y_P\})/h(\{x_P, y_P\})$ as a quotient of isobaric polynomials (with coefficients in the graded ring \mathcal{R}_1) in the variables $\{x_P, y_P \mid P \in E[\ell] - \{P_0\}\}$, where x_P has weight 2 and y_P has weight 3, so that the resulting weight of f is j ;
- (2) f satisfies an equation of graded integral dependence over the graded ring \mathcal{R}_1 . (Over \mathbf{C} , this requirement would ensure that we only select weight j elements of K_ℓ that are holomorphic at all $\tau \in \mathcal{H}$ and at all the cusps of the modular curve $X(\ell)$.)

Now in light of (3.6), we expect that the $\{x_P\}$ and $\{y_P\}$ will turn out to be modular forms of weights 2 and 3 by the above definition. We see that this is indeed the case for the $\{x_P\}$, since the equation of graded integral dependence that they satisfy is the square¹ of the ℓ -division polynomial $\psi_\ell^2(x) = \ell^2 \prod_{P \in E[\ell] - \{P_0\}} (x - x_P) = \ell^2 x^{\ell^2-1} + \dots \in \mathcal{R}_{1,\mathbf{Z}}[x]$ (see, for example, Exercise III.3.7 of [Sil86]). Similarly, the $\{y_P\}$ are integrally dependent over \mathcal{R}_1 by transitivity, using $y_P^2 = x_P^3 + ax_P + b$.

We note for later use a consequence of the above discussion. Since the division polynomial ψ_ℓ^2 is isobaric, the coefficient of x^{ℓ^2-2} is a weight 2 element of \mathcal{R}_1 , and must therefore vanish. This implies that

$$(3.13) \quad \sum_{P \in E[\ell] - \{P_0\}} x_P = 0.$$

Remark 3.3. Definition 3.2 implies that the graded ring of modular forms is the (graded) integral closure of \mathcal{R}_1 in K_ℓ . It is a pleasant exercise to verify that, over \mathbf{C} , this produces the usual graded ring of modular forms. (Part of the proof involves observing that K_ℓ contains a , b , and all the x_{PS} and y_{PS} , which, by Proposition 6.1 of [Shi71], suffice to generate the function field of $X(\ell)$ via weight 0 meromorphic ratios of elements of \mathcal{R}_ℓ .) We reassure the reader who wishes to avoid this verification that in any case we only use a certain subring \mathcal{R}_ℓ of the ring of modular forms, given in Definition 3.7, that is generated by Eisenstein series of weights ≤ 6 in case $k = \mathbf{C}$, and which turns out to be generated by the Eisenstein series of weight 1 (Theorems 3.9 and 3.13).

¹ We have used the square ψ_ℓ^2 here so as to avoid encountering a factor of y when ℓ is even; if we did not take the square, we would need to write $\psi_\ell(x, y)$ to allow for the presence of y .

Remark 3.4. The weight of a homogeneous element of K_ℓ can be defined intrinsically by considering, for each $u \in k^\times$, the automorphism of K_ℓ and corresponding isomorphism of elliptic curves given by:

$$(3.14) \quad \begin{aligned} a &\mapsto u^4 a, & b &\mapsto u^6 b, & \omega &\mapsto u^{-1} \omega, & t &\mapsto u^{-1} t, \\ (x, y) \in E : y^2 &= x^3 + ax + b &\mapsto (u^2 x, u^3 y) \in E' : y^2 &= x^3 + u^4 ax + u^6 b. \end{aligned}$$

This automorphism naturally sends $x_P \mapsto u^2 x_P$ and $y_P \mapsto u^3 y_P$, and is compatible with the grading on \mathcal{R}_1 ; hence a modular form f of weight j is sent by this automorphism to $u^j f$. Incidentally, $u = -1$ corresponds to inversion on E , since $\ominus P = (x_P, -y_P)$. This easily distinguishes modular forms of odd and even weight. In particular, we have

$$(3.15) \quad \lambda_{\ominus P} = -\lambda_P, \quad \mu_{\ominus P} = \mu_P, \quad x_{\ominus P} = x_P, \quad \nu_{\ominus P} = -\nu_P, \quad y_{\ominus P} = -y_P.$$

The above equations are for $P \in E[\ell] - \{P_0\}$. When $P = P_0$, we of course have $f_{P_0} = 1$, and so $\lambda_{P_0} = \mu_{P_0} = \nu_{P_0} = 0$. For convenience, we shall also define $x_{P_0} = y_{P_0} = 0$ in this case, even though the point P_0 , being at infinity, does not have affine coordinates. With this convention, we have the further identities

$$(3.16) \quad \sum_{P \in E[\ell]} \lambda_P = \sum_{P \in E[\ell]} \mu_P = \sum_{P \in E[\ell]} x_P = \sum_{P \in E[\ell]} \nu_P = \sum_{P \in E[\ell]} y_P = 0.$$

The above equations are obvious for the odd weights (λ_P, ν_P, y_P) , while $\sum x_P = 0$ is (3.13). We can however give a uniform proof of all of these results, including the fact that $\sum \mu_P = 0$. The motivation for the uniform proof is that each sum over all $P \in E[\ell]$ in (3.16) gives a modular form on $\Gamma(1)$ of weight 1, 2, or 3, which can only be zero. To see this algebraically, note that such a sum is invariant under the Galois group of the extension K_ℓ/K ; this group acts on the points of $E[\ell]$ in a way that preserves the Weil pairing (since $\mu_\ell \subset k$), and is easily seen to be isomorphic to $SL(2, \mathbf{Z}/\ell\mathbf{Z})$. Thus each such sum is a weight j element of $K = k(a, b)$ for some $j \in \{1, 2, 3\}$. Now by Theorem 3.9 below (the reader can check that no circular reasoning is involved), the above sums are all modular forms, and hence are integral over the subring $\mathcal{R}_1 = k[a, b]$. But \mathcal{R}_1 is integrally closed, and so the above sums actually belong to \mathcal{R}_1 , which means that they must vanish due to their weight.

We remark incidentally that an alternative proof of $\sum \mu_P = 0$ is contained in the proof of Proposition 4.3.

The following proposition is an easy consequence of the expansions in (3.8) and standard facts on elliptic curves:

Proposition 3.5. (1) *Let $P = (x_P, y_P) \in E[\ell] - \{P_0\}$. Then the divisor $(P) + (\ominus P) - 2(P_0)$ is principal, and we have*

$$(3.17) \quad \begin{aligned} f_{(P) + (\ominus P)} &= f_{(P) + (\ominus P) - 2(P_0)} = x - x_P \\ &= t^{-2}(1 - x_P t^2 - at^4 + \dots) \in t^{-2} \mathcal{R}_1[x_P][[t]]. \end{aligned}$$

In particular, $\lambda_{(P) + (\ominus P)} = \nu_{(P) + (\ominus P)} = 0$ and $\mu_{(P) + (\ominus P)} = -x_P$.

(2) *Let $P, Q, R \in E[\ell] - \{P_0\}$ satisfy $P \oplus Q \oplus R = P_0$; in other words, they are collinear in the affine Weierstrass model of E . Then $\lambda_{(P) + (Q) + (R)}$ is the slope of the line joining the three points. Specifically, the equation of the*

line is $y = \lambda_D x + \nu_D$, and we have the following for $D = (P) + (Q) + (R)$:

$$(3.18) \quad \begin{aligned} f_D &= f_{D-3(P_0)} = -y + \lambda_D x + \nu_D \\ &= t^{-3}(1 + \lambda_D t + \nu_D t^3 - at^4 + \cdots) \in t^{-3}\mathcal{R}_1[\lambda_D, \nu_D][[t]]. \end{aligned}$$

In particular, $\mu_D = 0$, and we have $\lambda_D = (y_P - y_Q)/(x_P - x_Q)$ if $P \neq Q$ (whence $x_P \neq x_Q$, since we cannot have $P = \ominus Q$ due to $R \neq P_0$). On the other hand, if $P = Q$, then $\lambda_D = (3x_P^2 + a)/2y_P$; here again, $y_P \neq 0$ since we again cannot have $P = Q \in E[2]$. In both cases, the following standard identity (whose first equality follows from (3.12)) shows that λ_D satisfies the integrality condition in part (2) of Definition 3.2, and is therefore a modular form of weight 1:

$$(3.19) \quad (\lambda_P + \lambda_Q + \lambda_R)^2 = \lambda_D^2 = x_P + x_Q + x_R.$$

Finally, we also record the trivial identity

$$(3.20) \quad \nu_D = y_P - \lambda_D x_P = y_Q - \lambda_D x_Q = y_R - \lambda_D x_R.$$

As mentioned before in Remark 3.3, we also include a direct proof in case $k = \mathbf{C}$ that the form λ_D in part (2) of the above proposition is a modular form:

Corollary 3.6. *If $k = \mathbf{C}$, then in part (2) of the above proposition take a principal lift $\tilde{D} = (\alpha) + (\beta) + (\gamma) - 3(0)$ of $D - 3(P_0)$. We then obtain*

$$(3.21) \quad \lambda_D(\tau) = -G_1(\tau, \alpha) - G_1(\tau, \beta) - G_1(\tau, \gamma) = -\zeta(\alpha) - \zeta(\beta) - \zeta(\gamma).$$

More generally, we have $\lambda_{D+E} = \lambda_D + \lambda_E$ from (2.17), and so for all D supported on $E[\ell]$, we conclude that

$$(3.22) \quad \lambda_D(\tau) = -G_1(\tau, D).$$

Proof. From (3.18), we have that $df_D/f_D = t^{-1}(-3 + \lambda_D t + \cdots)$. By (3.7), we know that t and z agree up to $O(z^4)$, so we obtain the desired result from (2.19) (recall that $G_1(\tau, 0) = 0$) and (2.13). The more general result now follows from (3.12). \square

We shall now define \mathcal{R}_ℓ for $\ell \geq 2$, generalizing our previous definition $\mathcal{R}_1 = k[a, b]$. Namely, we let \mathcal{R}_ℓ be the graded k -subalgebra of the ring of all modular forms on $\Gamma(\ell)$ that is generated by:

- The forms a and b , in weights 4 and 6,
- All coordinates x_P, y_P , in weights 2 and 3,
- All slopes λ_D for $D = (P) + (Q) + (R)$ as in Proposition 3.5, in weight 1.

(We do not need to include the ν_D in weight 3, since they already belong to \mathcal{R}_ℓ by (3.20).) In other words,

$$(3.23) \quad \mathcal{R}_\ell = k[a, b, \{x_P, y_P \mid P \in E[\ell] - \{P_0\}\}, \{\lambda_D \mid D = (P) + (Q) + (R) \sim 3P_0\}].$$

We easily have $\mathcal{R}_{\ell'} \subset \mathcal{R}_\ell$ for ℓ' a divisor of ℓ (including $\ell' = 1$).

We observe that the coefficients in the formal Laurent expansions (3.17) and (3.18) belong to \mathcal{R}_ℓ for all $\ell \geq 2$; moreover, the expansions in (3.17) and (3.18) respect the weights of the modular forms, in the sense that each series has the form $t^m(1 + c_1 t + c_2 t^2 + \cdots)$ where c_j is a modular form of weight j . This observation motivates the following definition:

Definition 3.7. Let \mathcal{R} be any graded subalgebra of the ring of modular forms (say on $\Gamma(\ell)$). An \mathcal{R} -balanced Laurent series in t is a series of the form

$$(3.24) \quad t^m \left(1 + \sum_{j=1}^{\infty} c_j t^j \right), \quad c_j \in \mathcal{R} \text{ of weight } j.$$

An analogous definition holds for series expressed in terms of the analytic uniformizer z (when k has characteristic zero); as the following elementary lemma observes, the condition of being \mathcal{R}_ℓ -balanced does not depend on whether one expands with respect to t or z .

Lemma 3.8. *Let \mathcal{R} be a graded algebra as above. Then*

- (1) *If $f(t)$ and $g(t)$ are \mathcal{R} -balanced Laurent series, then so are $f(t)g(t)$ and $f(t)/g(t)$.*
- (2) *If $f(t) = t^m(1 + c_1 t + \dots)$ is \mathcal{R} -balanced, with $\ell \mid m$, then the “principal branch” of the ℓ th root $f(t)^{1/\ell} = t^{m/\ell}(1 + c_1 t/\ell + \dots)$ is again \mathcal{R} -balanced.*
- (3) *Assume that k has characteristic 0. Then $z = z(t) = t + 2at^5/5 + \dots$ and $t = t(z) = z - 2az^5/5 + \dots$ are both \mathcal{R}_1 -balanced series. It follows that whenever $\mathcal{R}_1 \subset \mathcal{R}$, then a series $f(t)$ is \mathcal{R} -balanced if and only if $f(t(z))$ is.*
- (4) *If $f(t) = t^m(1 + c_1 t + \dots)$ is \mathcal{R} -balanced, then the logarithmic differential df/f has the expansion $df/f = t^{-1}(m + \sum_{j \geq 1} d_j t^j)dt$ with d_j a weight j element of \mathcal{R} .*

Proof. The first two assertions are elementary; for the second, recall that ℓ is invertible in k by assumption. The third follows because the invariant differential $\omega = dx/(2y) = dz$ has, by the first assertion, an \mathcal{R}_1 -balanced expansion $\omega = (1 + 2at^4 + \dots)dt$; now integrate to obtain that $z = z(t)$ is balanced. The rest is immediate. \square

We can now state the first main result of this section.

Theorem 3.9. (1) *Let D be a divisor supported on $E[\ell]$, as in Definition 3.1.*

Then $f_D(t)$ is an \mathcal{R}_ℓ -balanced Laurent series, and hence for all $j \geq 1$, $\lambda_D^{(j)}$ is a modular form of weight j on $\Gamma(\ell)$; furthermore, $\lambda_D^{(j)} \in \mathcal{R}_\ell$.

- (2) *The same result holds if we expand f_D with respect to the analytic uniformizer z in characteristic zero, as well as if we expand the logarithmic derivative df_D/f_D . Thus if $k = \mathbb{C}$, this theorem combined with Theorem 2.8 and Proposition 2.10 imply that all Eisenstein series on $\Gamma(\ell)$ belong to \mathcal{R}_ℓ .*

Proof. Proposition 3.5 already shows that $f_D(t)$ is \mathcal{R}_ℓ -balanced in the two cases (i) $D = (P) + (\ominus P)$ and (ii) $D = (P) + (Q) + (R)$ with $\oplus D = P_0$. A more general D that is supported on $E[\ell] - \{P_0\}$, but that still satisfies $\oplus D = P_0$, can be written as a \mathbb{Z} -linear combination of divisors D of types (i) and (ii). We can thus use the multiplicativity of the f_D from (2.17) and the first part of Lemma 3.8 to conclude that $f_D(t)$ is \mathcal{R}_ℓ -balanced in this case. For general D supported on $E[\ell]$, use if needed the second part of Lemma 3.8 to also conclude that $f_D(t)$ is \mathcal{R}_ℓ -balanced. The result for $f_D(z)$ is also immediate from the above lemma. \square

Remark 3.10. Statement (2) above can also be proved as in Sections 10.2-10.5 of [Shi07], by expressing the higher derivatives of \wp in terms of \wp , \wp' , and $a(\tau)$; this relates Eisenstein series of weights 4 and above to the forms x_P , y_P , and a .

For the cases $D = (P) + (\ominus P)$ and $D = (P) + (Q) + (R)$ as in Proposition 3.5, we note that the corresponding f_D are polynomials in x and y : namely, $f_D = x - x_P$ and $f_D = -y + \lambda_D x + \nu_D$. In this case, the value of f_D at a point $T \in E[\ell] - \{P_0\}$ is either $x_T - x_P$ or $-y_T + \lambda_D x_T + \nu_D$, which is a weight 2 or 3 element of \mathcal{R}_ℓ . More generally, we have the following result:

Corollary 3.11. *Assume in the setting of Theorem 3.9 that D is an effective divisor supported on $E[\ell] - \{P_0\}$, and assume that $\oplus D = P_0$, so that $D - (\deg D)(P_0)$ is principal. Then f_D is a polynomial in x and y , whose coefficients all belong to \mathcal{R}_ℓ . We can in fact expand*

(3.25)

$$\begin{aligned} f_D &= x^n - H_D^{(1)} x^{n-2} y + H_D^{(2)} x^{n-1} - H_D^{(3)} x^{n-3} y + \cdots, \text{ if } \deg D = 2n \geq 2; \\ f_D &= -x^n y + H_D^{(1)} x^{n+1} - H_D^{(2)} x^{n-1} y + H_D^{(3)} x^n + \cdots, \text{ if } \deg D = 2n + 3 \geq 3. \end{aligned}$$

(The choice of signs above ensures that the monomials $x^N = t^{-2N} + \cdots$ and $-x^N y = t^{-2N-3} + \cdots$ for varying N are normalized \mathcal{R}_1 -balanced Laurent series in t .) Moreover, $H_D^{(j)}$ is a weight j element of \mathcal{R} , and we have $H_D^{(j)} = \lambda_D^{(j)}$ for $j = 1, 2, 3$, but not for $j \geq 4$. Finally, for every $T \in E[\ell] - \{P_0\}$, we have $f_D(T)$ is an element of \mathcal{R}_ℓ of weight $\deg D$.

Proof. The coefficients $H_D^{(j)}$ above can be computed from the Laurent expansion of f_D by successively subtracting multiples of x^N and $-x^N y$ for N going from n down to 0. \square

Remark 3.12. We do not use the results of the above corollary in this article, but we anticipate that they will be useful in other places. For example, the translation $\tau_T^* f_D$ of a function f_D by an element $T \in E[\ell]$ has as divisor the translation $D' = \tau_{\ominus T}(D)$ of D by $\ominus T$; here $\tau_T^* f_D$ will not be normalized, but if T does not belong to the support of D we can still write $\tau_T^* f_D = f_D(T) f_{D'}$ and deduce useful formulas. Another interesting example is the case when we take a principal divisor $D = (P_1) + (P_2) + (P_3) + (P_4) - 4(P_0)$, such that $D \neq \ominus D$. Then $f_D = x^2 - \lambda_D y + \mu_D x + H_D^{(4)}$, and λ_D cannot equal zero because D is not an “even” divisor. This means that, over \mathbf{C} , λ_D is then a weight 1 modular form that cannot vanish at any point of \mathcal{H} , but that can only vanish at the cusps; hence it is a kind of generalized modular unit constructed from weight 1 Eisenstein series. A simple example of this is the case $P_0 = P_1 = P$, $P_2 = \ominus P \ominus Q$, and $P_3 = \ominus P \oplus Q$, for $P \notin E[2] \cup \{Q, \ominus Q\}$. In this case we obtain $\lambda_D = \lambda_{(P)+(Q)+(\ominus P \ominus Q)} + \lambda_{(P)+(\ominus Q)+(\ominus P \oplus Q)} = 2y_P/(x_P - x_Q)$; this expression appears below in the proof of Theorem 3.13. The numerator and denominator in this expression are modular forms that are well known to vanish only at the cusps, as seen in [KL81]. Our methods have just shown that the ratio of these two forms is also a modular form (i.e., it does not have any poles, even at the cusps), and that this ratio is in fact λ_D , an Eisenstein series of weight 1.

Our second main result in this section is the fact that \mathcal{R}_ℓ is generated by its elements $\{\lambda_P\}$ of weight 1, in other words (over \mathbf{C}) by Eisenstein series of weight 1. This result holds only for $\ell \geq 3$. Indeed, if $\ell = 2$, then write as usual $E[2] = \{P_0, P_1, P_2, P_3\}$ with $P_i = (e_i, 0)$ for $1 \leq i \leq 3$. Hence $x_{P_i} = e_i$ and $y_{P_i} = 0$ for $1 \leq i \leq 3$, and all the λ_P are zero in this case; moreover, $(x - e_1)(x - e_2)(x - e_3) = x^3 + ax + b$, as usual. We easily obtain that \mathcal{R}_2 is the full ring of modular forms on

$\Gamma(2)$, namely $\mathcal{R}_2 = k[e_1, e_2, e_3 \mid e_1 + e_2 + e_3 = 0]$, which is generated by Eisenstein series of weight 2 when $k = \mathbf{C}$.

Theorem 3.13. *Assume that $\ell \geq 3$. Define the subring \mathcal{R}' of \mathcal{R}_ℓ to be the subring generated by all λ_D , where $D = (P) + (Q) + (R)$ is a divisor supported on $E[\ell] - \{P_0\}$ with $\oplus D = P_0$ as in part (2) of Proposition 3.5. Then the forms $a, b, \{x_P\}, \{y_P\}$, for $P \in E[\ell] - \{P_0\}$, all belong to \mathcal{R}' . In particular, $\mathcal{R}_\ell = \mathcal{R}'$ and is hence generated by the λ_D of the above form.*

Proof. We begin by showing that all the $\{x_P\}$ belong to \mathcal{R}' . This boils down to a judicious use of (3.19), and involves three cases, depending on ℓ :

- (1) If $\ell \geq 5$, let P be a point of exact order ℓ , and consider the following four elements of \mathcal{R}' (recall also that $x_{\ominus P} = x_P$):

$$(3.26) \quad \begin{aligned} (\lambda_{(P)+(P)+([-2]P)})^2 &= x_P + x_P + x_{[-2]P} = 2x_P + x_{[2]P} \\ (\lambda_{(P)+([2]P)+([-3]P)})^2 &= x_P + x_{[2]P} + x_{[-3]P} = x_P + x_{[2]P} + x_{[3]P} \\ (\lambda_{(P)+([3]P)+([-4]P)})^2 &= x_P + x_{[3]P} + x_{[4]P} \\ (\lambda_{([2]P)+([2]P)+([-4]P)})^2 &= 2x_{[2]P} + x_{[4]P}. \end{aligned}$$

Here the determinant $\det \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 1 \end{pmatrix} = 6$ is invertible in k , and so each

of $x_P, x_{[2]P}, x_{[3]P}, x_{[4]P}$ can be expressed in terms of λ 's, and so belongs to \mathcal{R}' . Now if $P \in E[\ell]$ is a point of order less than ℓ , we can find a basis $\{Q, R\}$ for $E[\ell] \cong (\mathbf{Z}/\ell\mathbf{Z})^2$, such that $P = [d]Q$ for some $d > 1$. In that case, the points $P' = (\ominus P) \oplus R = [-d]Q \oplus R$ and $P'' = \ominus R$ both have exact order ℓ , so $x_{P'}$ and $x_{P''}$ both belong to \mathcal{R}' . The points P, P', P'' are collinear, and so $(\lambda_{(P)+(P')+(P'')})^2 = x_P + x_{P'} + x_{P''}$ belongs to \mathcal{R}' , whence $x_P \in \mathcal{R}'$. (Alternatively, we can deal with a point $P = [d]Q$ of order less than ℓ by using identities analogous to (3.26) to see that $x_Q + x_{[n]Q} + x_{[n+1]Q} \in \mathcal{R}'$, and to deduce inductively that the x -coordinates of all multiples $[n]Q$ belong to \mathcal{R}' whenever Q has exact order ℓ .)

- (2) If $\ell = 3$, we simply note that $(\lambda_{3(P)})^2 = 3x_P$ for all $P \in E[3] - \{P_0\}$.
(3) If $\ell = 4$, let $\{Q, R\}$ be a basis for $E[4] \cong (\mathbf{Z}/4\mathbf{Z})^2$. By the same technique as in the first case above, we see that the following sums belong to \mathcal{R}' , being squares of suitable λ 's:

$$(3.27) \quad \begin{array}{llll} 2x_Q & +x_{[2]Q}, & & \\ & 2x_R & +x_{[2]R}, & \\ x_Q & +x_R & +x_{Q \oplus R}, & \\ x_Q & +x_R & & +x_{Q \oplus \ominus R}, \\ & x_{[2]Q} & +x_{Q \oplus R} & +x_{Q \oplus \ominus R}, \\ & & x_{[2]R} & +x_{Q \oplus R} & +x_{Q \oplus \ominus R}. \end{array}$$

(For example, the fourth sum above is $(\lambda_{(Q)+(\ominus R)+(R \oplus Q)})^2$.) The corresponding determinant is -12 , again invertible, so we deduce in particular that $x_Q, x_{[2]Q} \in \mathcal{R}'$. Now any $P \in E[4] - \{P_0\}$ has exact order either 4 or 2. So we can choose our basis $\{Q, R\}$ so as to have $P = Q$ in the former case, and $P = [2]Q$ in the latter case, thereby concluding that $x_P \in \mathcal{R}'$.

Now that we have shown that all the x_P belong to \mathcal{R}' , let us show that all the y_P also belong to \mathcal{R}' . Fix $P \in E[\ell] - \{P_0\}$, and take any $Q \in E[\ell] - \{P_0, P, \ominus P\}$. Then $(y_P - y_Q)/(x_P - x_Q)$ and $(y_P + y_Q)/(x_P - x_Q)$ are among our λ 's (the latter being the slope of the line through P and $\ominus Q$), and so their sum $2y_P/(x_P - x_Q)$ belongs to \mathcal{R}' . Multiplying by $x_P - x_Q \in \mathcal{R}'$ shows that $y_P \in \mathcal{R}'$. Observe that at this point we know by (3.20) that the forms $\{\nu_D\}$ for $D = (P) + (Q) + (R)$ also belong to \mathcal{R}' .

Finally, take any $P \in E[\ell] - E[2]$. Then $a = 2y_P\lambda_{(P)+(P)+([-2]P)} - 3x_P^2$ also belongs to \mathcal{R}' , as does $b = y_P^2 - x_P^3 - ax_P$. Alternatively, we can deduce that $a, b \in \mathcal{R}'$ from the polynomial identity $(x - x_P)(x - x_Q)(x - x_R) = x^3 - (\lambda_D x + \nu_D)^2 + ax + b$ whenever $D = (P) + (Q) + (R)$ with P, Q, R collinear as usual. \square

Remark 3.14. We can also define a subring \mathcal{R}'_A of \mathcal{R}' , corresponding to a subgroup $A \subset E[\ell]$: let \mathcal{R}'_A be generated by the forms $\lambda_{(P)+(Q)+(R)}$, for $P, Q, R \in A - \{P_0\}$ with $P \oplus Q \oplus R = P_0$. Assume that $A \cong \mathbf{Z}/m\mathbf{Z} \oplus \mathbf{Z}/\ell\mathbf{Z}$ with $m|\ell$ and $\ell \geq 5$ (possibly $m = 1$). Then our methods of proof show that $a, b, \{x_P, y_P \mid P \in A - \{P_0\}\}$ all belong to \mathcal{R}'_A , as do the appropriate ν 's coming from points in A . Compare this to Proposition 4.9 in [BG01a].

Remark 3.15. The above two theorems show that when $\ell \geq 3$, all the modular forms that we have constructed through Laurent expansions can be expressed as polynomials in the λ_D , which are special Eisenstein series of weight 1 when $k = \mathbf{C}$. It is equally useful to consider a different set of generators of \mathcal{R}_ℓ , namely the $\{\lambda_P \mid P \in E[\ell] - \{P_0\}\}$. We have the relation $\lambda_{(P)+(Q)+(R)} = \lambda_P + \lambda_Q + \lambda_R$, which shows that the $\{\lambda_P\}$ for single points generate the $\{\lambda_D\}$ as above. Our proof above gives a rather indirect proof of the converse statement, that the $\{\lambda_D\}$ generate the $\{\lambda_P\}$. One can also see this converse directly by observing that ℓ is invertible in k and that $\ell\lambda_P = \sum_{n=1}^{\ell-2} \lambda_{(P)+([n]P)+([-n-1]P)}$. Alternatively, one can express λ_P as a linear combination of $O(\log \ell)$ different λ_D s using values of n starting from 1 and increasing by a “double-and-add” approach until we reach $n = \ell - 1$. This is left to the reader.

We conclude this section by noting a couple of useful algebraic relations between the modular forms in \mathcal{R}_ℓ . We note that (3.30) below has already appeared for $\Gamma_1(\ell)$ in [BG01b, BGP01]. The approach of obtaining relations by taking a sum of residues over all points of E is taken from [BG01a].

Lemma 3.16. (1) *Let $P \in E[\ell] - \{P_0\}$. Then the Laurent expansion of the logarithmic differential df_P/f_P begins with*

$$(3.28) \quad df_P/f_P = t^{-1}[-1 + \lambda_P t - x_P t^2 + y_P t^3 + \dots]dt.$$

(This is the algebraic analog of (2.21), taking into account (3.6), (3.7), and (3.22).) We deduce the following equations, which over \mathbf{C} can also be seen from (2.23):

$$(3.29) \quad x_P = \lambda_P^2 - 2\mu_P, \quad y_P = 3\nu_P - 3\mu_P\lambda_P + \lambda_P^3.$$

(2) *Let $D = (P) + (Q) + (R)$ be as usual a divisor supported on $E[\ell] - \{P_0\}$ with $\oplus D = P_0$. Then*

$$(3.30) \quad \lambda_P\lambda_Q + \lambda_Q\lambda_R + \lambda_P\lambda_R + \mu_P + \mu_Q + \mu_R = 0.$$

Proof. For (3.28) and (3.29), consider the meromorphic differential form df_P/f_P on E . Recall that $f_P = (f_{\ell(P)-\ell(P_0)})^{1/\ell}$ exists in $\hat{\mathcal{O}}$ but is not a meromorphic function on E ; however, its logarithmic differential is globally defined since $f_{\ell(P)-\ell(P_0)}$ is a global meromorphic function on E , and df_P/f_P has simple poles at each of P_0 and P , with residues -1 and 1 , respectively. Now use the fact that the sum of the residues of the global meromorphic differential $x df_P/f_P$ (respectively, $y df_P/f_P$) at all points of $E(\bar{K})$ is zero. Taking into account the fact that $x = t^{-2}(1 + O(t^4))$ and $y = -t^{-3}(1 + O(t^4))$, this yields the coefficients x_P and y_P in (3.28). On the other hand, we can directly compute the logarithmic differential of $f_P = t^{-1}(1 + \lambda_P t + \mu_P t^2 + \nu_P t^3 + \dots)$, and this yields the coefficient λ_P in (3.28), as well as (3.29). Finally, to see (3.30), combine the equations $x_P = \lambda_P^2 - 2\mu_P$ for P , Q , and R with (3.19). \square

4. RELATIONS INVOLVING THE WEIL PAIRING AND HECKE OPERATORS

In this section, we prove deeper algebraic relations between the modular forms $\lambda_D^{(j)}$ than those in Lemma 3.16. The first few relations owe their existence to the Weil pairing on the ℓ -torsion group $E[\ell]$ of our elliptic curve. Others are related to the action of the full Hecke algebra of $\Gamma(\ell)$ on modular forms in \mathcal{R}_ℓ . We eventually obtain enough relations to be able to show in essence that, over \mathbf{C} , the weight 2 and 3 parts of \mathcal{R}_ℓ are stable under the action of the Hecke algebra. (Actually, in the case of weight 3 we obtain only a partial result at this stage of the proof.) We use this in Section 5 to conclude over \mathbf{C} that the ring \mathcal{R}_ℓ contains all modular forms of weights 2 and above. This of course implies Hecke stability in all weights, and supersedes the previous result. Thus the only modular forms that do not appear in \mathcal{R}_ℓ are the cusp forms of weight 1; all other modular forms of all weights are expressible as polynomials in the λ_P , or equivalently as polynomials in the $\lambda_{(P)+(Q)+(R)}$ which are slopes of lines through torsion points of the Weierstrass model of E .

The overall shape of the formulas giving the action of the Hecke operators is similar to the results in the articles of Borisov and Gunnells [BG01a, BG01b, BG03]. The treatment in those articles concerns only the group $\Gamma_1(\ell)$, and proceeds via q -expansions and periods of modular forms (the reader is referred also to [Paş06]). Our formulation in terms of $\Gamma(\ell)$ involves neither of the above techniques, but focuses instead on the modular parametrization given by the modular curve. We hope to treat some of the connections between this article and those previous articles in later work; it would also be desirable to understand the Hecke action better by directly relating our relations coming from Laurent expansions of elliptic functions to the geometry of toric varieties used in [BG01a].

In order to introduce the Weil pairing on $E[\ell]$, we also need to discuss pullbacks (i.e., composition) of elements $\hat{\mathcal{O}}$ by the multiplication map $[n] : E \rightarrow E$; our main concern is to define the element $f_Q \circ [n] \in \hat{\mathcal{O}}$, in the sense of controlling its algebraic Laurent expansion in terms of t . This can be done entirely inside the formal group, since we have an expansion of the form $t \circ [n] = nt + 2at^5(n - n^5)/5 + O(t^7) \in \mathcal{R}_{1,\mathbf{Z}}[[t]]$, so we can formally obtain $f_Q \circ [n] = n^{-1}t^{-1}(1 + \lambda_Q nt + \dots)$. At the same time, we can identify $f_Q \circ [n]$ by its formal zeros and poles as below, and normalize it by a constant factor so that its expansion begins with $n^{-1}t^{-1}$. We thus obtain the first part of the following definition.

Definition 4.1. (1) Let $Q \in E[\ell] - \{P_0\}$ and let $1 \leq n \in \mathbf{Z}$, with n invertible in k . Choose a point $Q' \in E[n\ell]$ such that $[n]Q' = Q$. Then define the element $f_Q \circ [n] := n^{-1}f_D \in \hat{\mathcal{O}}$, where $D = \sum_{T \in E[n]}(Q' \oplus T) - \sum_{T \in E[n]}(T)$. Here $f_Q \circ [n]$ is usually not an element of the function field of E , but we have the Laurent expansion

$$(4.1) \quad f_Q \circ [n] = n^{-1}t^{-1}(1 + \lambda_Q n t + \mu_Q n^2 t^2 + \nu_Q n^3 t^3 + O(t^4)),$$

where the remaining terms after t^3 do *not* follow the simple initial pattern. (In case $Q = P_0$, we have $f_{P_0} = f_{P_0} \circ [n] = 1$.)

- (2) In the special case $n = \ell$, we introduce the notation $g_Q = f_Q \circ [\ell]$. In this setting, g_Q is a genuine element of $K_\ell(E)$, since the divisor D of part (1) is now principal.
- (3) The Weil pairing $e_\ell : E[\ell] \times E[\ell] \rightarrow \mu_\ell$ is given (as usual) by the behavior of the functions g_Q under translation by elements of $E[\ell]$: namely,

$$(4.2) \quad g_Q(P \oplus R) = e_\ell(Q, R)g_Q(P), \text{ where } Q, R \in E[\ell] \text{ and } P \in E(\overline{K}).$$

Remark 4.2. If $k = \mathbf{C}$, consider the case when $Q = P_{1/\ell}$ and $R = P_{\tau/\ell}$. One can then show that our normalization gives $e_\ell(P_{1/\ell}, P_{\tau/\ell}) = e^{2\pi i/\ell}$. (The easiest way to do this calculation is to avoid the Weierstrass σ -function; instead, begin by showing that $g_{P_{1/\ell}}(z) = C \cdot \vartheta(\ell z - 1/\ell)/\vartheta(\ell z)$ for some nonzero constant C , where $\vartheta = \vartheta_{11}$.)

We are now ready for the relations arising from the Weil pairing. In weight 1, they imply a subtle symmetry between the $\{\lambda_P\}$, essentially a duality under the Fourier transform on $E[\ell]$ induced by the pairing e_ℓ . When $k = \mathbf{C}$, this subtle symmetry motivates Hecke's result that the dimension of the space of Eisenstein series of weight 1 on $\Gamma(\ell)$ is half the number of cusps of $X(\ell)$ (see the end of Section 2 of [Hec27]). This symmetry is usually expressed in terms of q -expansions of weight 1 Eisenstein series; see the second identity at the beginning of Section 7 of [Hec26], which is also derived in Sections 3.4 and 3.5 of [Kat76].

Proposition 4.3. *The following identities hold for all $R \in E[\ell]$, where we use the conventions of Remark 3.4 (thus the sums over Q below are unchanged if we sum instead over $Q \in E[\ell] - \{P_0\}$):*

$$(4.3) \quad \begin{aligned} \lambda_R &= \frac{-1}{\ell} \sum_{Q \in E[\ell]} \lambda_Q e_\ell(Q, R), \\ x_R &= - \sum_{Q \in E[\ell]} \mu_Q e_\ell(Q, R), \\ y_R &= -\ell \sum_{Q \in E[\ell]} \nu_Q e_\ell(Q, R). \end{aligned}$$

By Fourier inversion on the finite group $E[\ell]$, we obtain from (4.3) the identities

$$(4.4) \quad \mu_R = \frac{-1}{\ell^2} \sum_{Q \in E[\ell]} x_Q e_\ell(Q, R), \quad \nu_R = \frac{-1}{\ell^3} \sum_{Q \in E[\ell]} y_Q e_\ell(Q, R).$$

Proof. Let $Q \in E[\ell] - \{P_0\}$, and consider the meromorphic function $g_Q = f_Q \circ [\ell] \in K_\ell(E)$, whose Laurent expansion is of the form (4.1). Define the global meromorphic differential form $\eta_Q = g_Q \omega$ on E , where $\omega = (1 + O(t^4))dt$ is the invariant differential; the only singularities of η_Q are simple poles at the points of

$E[\ell]$. Now the residue of η_Q at P_0 is ℓ^{-1} , and (4.2) says that $\tau_R^* \eta_Q = e_\ell(Q, R) \eta_Q$, where $\tau_R : E \rightarrow E$ is translation by R . Thus the residue of η_Q at any $R \in E[\ell]$ is $\ell^{-1} e_\ell(Q, R)$. Now define the differential form $\eta = -\ell \sum_{Q \in E[\ell] - \{P_0\}} \eta_Q$. We see that η has simple poles at all the points of $E[\ell]$, and that the residue of η at P_0 is $-\ell^2 + 1$, while the residue at $R \in E[\ell] - \{P_0\}$ is 1, by the nondegeneracy of the Weil pairing. More precisely, the series expansions of η and $\tau_R^* \eta$ for $R \neq P_0$ have the following form (the sums below are over $Q \in E[\ell] - \{P_0\}$):

$$\begin{aligned}
 (4.5) \quad \eta &= t^{-1} \left[(-\ell^2 + 1) + \sum_Q \lambda_Q \ell t + \sum_Q \mu_Q \ell^2 t^2 + \sum_Q \nu_Q \ell^3 t^3 + \cdots \right] dt \\
 &= t^{-1} \left[(-\ell^2 + 1) + O(t^4) \right] dt, \\
 \tau_R^* \eta &= t^{-1} \left[1 - \sum_Q \lambda_Q e_\ell(Q, R) \ell t - \sum_Q \mu_Q e_\ell(Q, R) \ell^2 t^2 - \sum_Q \nu_Q e_\ell(Q, R) \ell^3 t^3 + \cdots \right] dt.
 \end{aligned}$$

The second equality above in the expansion of η follows from (3.16) (which incidentally yields (4.3) in the special case $R = P_0$; see however the upcoming footnote in this proof). The expansion of $\tau_R^* \eta$ holds because $\tau_R^* \eta = -\ell \sum_Q e_\ell(Q, R) \eta_Q$.

We now relate the differential form η to the function f_D , corresponding to the divisor $D = \sum_{Q \in E[\ell]} (Q)$. The divisor of f_D is $(f_D) = \left(\sum_{Q \in E[\ell]} (Q) \right) - \ell^2 (P_0) = \left(\sum_{Q \in E[\ell] - \{P_0\}} (Q) \right) + (-\ell^2 + 1)(P_0)$. Thus η and df_D/f_D have poles at the same locations, with the same residues. We claim that in fact $\eta = df_D/f_D$, since the difference is not only everywhere holomorphic, but also vanishes at P_0 , by looking beyond the first term in the Laurent expansions at P_0 . Indeed, $f_D = \pm \ell^{-1} \psi_\ell(x, y)$ where ψ_ℓ is the ℓ th division polynomial, and hence f_D has an \mathcal{R}_1 -balanced Laurent expansion of the form $f_D = t^{-\ell^2+1}(1 + O(t^4))$, which implies that $df_D/f_D = t^{-1}[(-\ell^2 + 1) + O(t^4)]$; on the other hand, η has a similar expansion by (4.5), and our claim follows.²

We now consider the translation of the identity $\eta = df_D/f_D$ by the point R , when $R \neq P_0$. This gives us $\tau_R^* \eta = \tau_R^* (df_D/f_D) = d(\tau_R^* f_D)/\tau_R^* f_D$. We shall compare the expansion of $\tau_R^* \eta$ from (4.5) to the expansion of the logarithmic differential of $\tau_R^* f_D$. Comparing the locations of zeros and poles, we see that $\tau_R^* f_D = C \cdot f_D \cdot (f_{(\ominus R)})^{-\ell^2}$ for some nonzero constant C . (Here $f_{(\ominus R)}$ is not a genuine meromorphic function on E , but its ℓ th power is, so $(f_{(\ominus R)})^{-\ell^2}$ is also a genuine meromorphic function.) We obtain that $d(\tau_R^* f_D)/\tau_R^* f_D = df_D/f_D - \ell^2 df_{\ominus R}/f_{\ominus R}$. However, from (3.28) and (3.15), we have

$$(4.6) \quad df_{\ominus R}/f_{\ominus R} = t^{-1}[-1 - \lambda_R t - x_R t^2 - y_R t^3 + \cdots] dt.$$

Combining all this and comparing the Laurent expansions, we obtain (4.3) as desired. Equation (4.4) then follows immediately. \square

² The alert reader will note that we needed only the simple result $\sum_Q \lambda_Q = 0$ of (3.16) to deduce that $\eta = df_D/f_D$. The form of the expansion of f_D then allows us to conclude the identity $\sum_Q \mu_Q = 0$ — as well as the simple identity $\sum_Q \nu_Q = 0$ — thereby giving a second way to complete the proof of (3.16), and hence of (4.3) when $R = P_0$.

The relations (4.4), when combined with (3.6), imply that the forms $\{\mu_P, \nu_P\}$ are Eisenstein series of weights 2 and 3, when $k = \mathbf{C}$. It will be useful for us to formalize this algebraically, while also taking into account (3.22).

Definition 4.4. For $j \in \{1, 2, 3\}$, we define the algebraic space \mathcal{E}_j of Eisenstein series of weight j by

$$(4.7) \quad \mathcal{E}_1 = \text{span}\{\lambda_P \mid P \in E[\ell]\}, \quad \mathcal{E}_2 = \text{span}\{x_P\}, \quad \mathcal{E}_3 = \text{span}\{y_P\}.$$

(If we wish to draw attention to the level ℓ , we will write \mathcal{E}_j^ℓ .)

We deduce from (4.4) and (3.29) that for all $P \in E[\ell]$,

$$(4.8) \quad \mu_P, \lambda_P^2 \in \mathcal{E}_2, \quad \nu_P \in \mathcal{E}_3.$$

From (3.30), we also obtain that for $P, Q, R \in E[\ell]$ with $P \oplus Q \oplus R = P_0$,

$$(4.9) \quad \lambda_P \lambda_Q + \lambda_Q \lambda_R + \lambda_P \lambda_R \in \mathcal{E}_2.$$

Note that in the above equation, the points P, Q, R are allowed to take the value P_0 ; for example, if $Q = P_0$, then $\lambda_R = -\lambda_P$, in which case (4.9) becomes the statement $-\lambda_P^2 \in \mathcal{E}_2$ that we know from (4.8). (The result that μ_P and λ_P^2 are Eisenstein series, as well as the result (4.9), were already observed for $\Gamma_1(\ell)$ in [BG01b]).

In our treatment of Hecke operators, we shall need the following identities, which are related to the fact that the trace from $\Gamma(n\ell)$ to $\Gamma(\ell)$ of an Eisenstein series on $\Gamma(n\ell)$ is again an Eisenstein series.

Lemma 4.5. *Let $n \geq 1$ be invertible in k . Let $P \in E[n\ell]$ (typically, $P \in E[\ell]$), and let $T \in E[n]$. Consider the modular forms $\lambda_{P \oplus T}$, $x_{P \oplus T}$, and $y_{P \oplus T}$ on $\Gamma(n\ell)$. We then have*

$$(4.10) \quad \sum_{T \in E[n]} \lambda_{P \oplus T} = n \lambda_{[n]P}, \quad \sum_{T \in E[n]} x_{P \oplus T} = n^2 x_{[n]P}, \quad \sum_{T \in E[n]} y_{P \oplus T} = n^3 y_{[n]P}.$$

We also have

$$(4.11) \quad \sum_{T \in E[n]} \mu_{P \oplus T} = \mu_{[n]P}, \quad \sum_{T \in E[n]} \nu_{P \oplus T} = \frac{1}{n} \nu_{[n]P}.$$

Proof. Over \mathbf{C} , equation (4.10) is immediate from the definition of G_j in (2.1) and (2.2) — in verifying this, the reader should bear in mind that x_P is a difference between two G_2 s. Let us however give a proof in our algebraic setting. Now (4.10) is trivial for $P = P_0$. If $P \neq P_0$, we begin by noting the following identity, which is obtained by comparing zeros and poles, as well as the leading coefficient of the Laurent expansion:

$$(4.12) \quad f_{[n]P} \circ [n] = n^{-1} \left(\prod_{T \in E[n]} f_{P \oplus T} \right) / f_D.$$

Here $D = \sum_{T \in E[n]} (T)$ is the divisor supported on the n -torsion points. Hence the principal divisor of f_D is $(f_D) = D - n^2(P_0)$, similarly to the proof of Proposition 4.3; this also implies the expansion $f_D = t^{-n^2+1}(1 + O(t^4))$. Now taking the logarithmic differential of both sides of (4.12) and comparing the first few coefficients yields (4.10), as desired.

As for (4.11), we prove it using the Fourier duality of Proposition 4.3. (This approach also yields a different proof of (4.10).) For instance, use (4.4) to express each μ in the first sum in (4.11) in terms of an x . This yields

$$(4.13) \quad \sum_{T \in E[n]} \mu_{P \oplus T} = \sum_{T \in E[n]} \frac{-1}{n^2 \ell^2} \sum_{A \in E[n\ell]} x_A e_{n\ell}(A, P \oplus T).$$

Rearrange the sum as $\sum_A \sum_T$, and use the property of the Weil pairing

$$(4.14) \quad A \in E[n\ell], \quad T \in E[n] \implies e_{n\ell}(A, T) = e_n([\ell]A, T)$$

to conclude that the only surviving terms are those when $[\ell]A = P_0$, in other words, for $A \in E[\ell]$. Thus we obtain

$$(4.15) \quad \sum_{T \in E[n]} \mu_{P \oplus T} = \frac{-n^2}{n^2 \ell^2} \sum_{A \in E[\ell]} x_A e_{n\ell}(A, P) = \frac{-1}{\ell^2} \sum_{A \in E[\ell]} x_A e_\ell(A, [n]P),$$

where the last equality is analogous to (4.14). This implies the first part of (4.11). The second part, involving ν , is proved similarly. \square

We are now ready for the main ingredient in the proof that the degree 2 part of \mathcal{R}_ℓ is stable under the Hecke algebra. This proof involves an interesting induction on the level. One starts with forms on $\Gamma(n\ell)$, “raises the level” to rewrite them in terms of forms of higher level $\Gamma(sn\ell)$ with $s < n$, then “lowers the level” back to level $\Gamma(s\ell)$. Repeating this process reduces the value of s , and one eventually reaches $s = 0$, which can be dealt with using Lemma 4.5.

Proposition 4.6. *Let $n \geq 1$ and assume that $n!$ is invertible in k . Let $A, B \in E[n\ell]$ (as before, typically $A, B \in E[\ell]$), and let $s \in \mathbf{Z}$. Then*

$$(4.16) \quad \begin{aligned} & \sum_{T \in E[n]} \lambda_{A \oplus T} \lambda_{B \ominus [s]T} \\ &= (a \text{ linear combination of terms of the form } \lambda_{[a]A \oplus [b]B} \lambda_{[c]A \oplus [d]B}) \\ & \quad + (\text{an element of } \mathcal{E}_2^{n!\ell}), \end{aligned}$$

where the linear combination above is over finitely many $(a, b, c, d) \in \mathbf{Z}^4$ satisfying

$$(4.17) \quad \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm n, \quad a - sb \equiv c - sd \equiv 0 \pmod{n}.$$

Proof. The proof is by induction on n , the case $n = 1$ (so $T = P_0$) being trivial. Note that the value of s only matters modulo n , so we henceforth assume that $0 \leq s < n$. If $s = 0$, then the sum over T is $n\lambda_{[n]A}\lambda_B$ by (4.10), so we are done. In general, we shall invoke an inductive step analogous to the Euclidean algorithm, reducing (4.16) for the pair (n, s) to the analogous statement for (s, n) , which amounts to the same as $(s, n \bmod s)$. To this end, choose a point $B' \in E[sn\ell]$ for which $[s]B' = B$. We then see from (4.10) that

$$(4.18) \quad \lambda_{B \ominus [s]T} = s^{-1} \sum_{U \in E[s]} \lambda_{B' \ominus T \oplus U}.$$

Hence, up to the factor s^{-1} , our sum in (4.16) becomes

$$(4.19) \quad \sum_{T \in E[n], U \in E[s]} \lambda_{A \oplus T} \lambda_{B' \ominus T \oplus U} \\ \equiv \sum_{T, U} \lambda_{A \oplus T} \lambda_{A \oplus B' \oplus U} - \sum_{T, U} \lambda_{\ominus B' \oplus T \oplus U} \lambda_{A \oplus B' \oplus U} \pmod{\mathcal{E}_2^{sn\ell}},$$

where the congruence is obtained from (4.9) with $P = A \oplus T$, $Q = B' \ominus T \oplus U$, and $R = \ominus A \oplus B' \ominus U$; we have also used (3.15). Now the first sum on the right hand side of equation (4.19) is a constant (namely, ns) times $\lambda_{[n]A} \lambda_{[s](A \oplus B')} = \lambda_{[n]A} \lambda_{[s]A \oplus B}$, which has the desired form. On the other hand, the second sum on the right hand side can be summed first over all $T \in E[n]$, which by (4.10) yields a constant times

$$(4.20) \quad \sum_{U \in E[s]} \lambda_{[-n]B' \ominus [n]U} \lambda_{A \oplus B' \oplus U}.$$

By the inductive hypothesis, the above sum is congruent modulo $\mathcal{E}_2^{s!n\ell}$ to a linear combination of terms of the form

$$(4.21) \quad \lambda_{[a'](A \oplus B') \oplus [-nb']B'} \lambda_{[c'](A \oplus B') \oplus [-nd']B'} = \lambda_{[a']A \oplus [\frac{a'-nb'}{s}]B} \lambda_{[c']A \oplus [\frac{c'-nd'}{s}]B}$$

where (a', b', c', d') satisfy (4.17) with the roles of s and n interchanged; in particular, $\frac{a'-nb'}{s}, \frac{c'-nd'}{s} \in \mathbf{Z}$, and we get that each term is of the form $\lambda_{[a]A \oplus [b]B} \lambda_{[c]A \oplus [d]B}$, satisfying the original requirements of (4.17). Finally, we remark that $\mathcal{E}_2^{sn\ell}$ and $\mathcal{E}_2^{s!n\ell}$ are both subspaces of $\mathcal{E}_2^{n!l}$. \square

Remark 4.7. The element of $\mathcal{E}_2^{n!l}$ above actually belongs to $\mathcal{E}_2^{n\ell}$, but we shall not prove this in our algebraic context; it is obvious over \mathbf{C} , since it is an Eisenstein series with level $n!l$ that happens to transform under $\Gamma(n\ell)$. (Similarly, if $A, B \in E[\ell]$, then the element of \mathcal{E}_2 above actually belongs to \mathcal{E}_2^ℓ .) It is possible to specify this element more precisely by applying (3.30) instead of (4.9) in the above proof. Typically, this yields an element of \mathcal{E}_2 that is a linear combination of terms $\mu_{[a]A + [b]B}$ where $a - sb \equiv 0 \pmod{n}$, after one also invokes (4.11). However, one must be careful not to apply (3.30) when one of the torsion points P, Q, R is P_0 .

On another topic, we observe that the linear combination in (4.16) is \mathbf{Z} -linear, and the coefficients are all divisible by n . This we leave to the reader.

We need a few more standard observations before we prove the Hecke stability of the weight 2 part of \mathcal{R}_ℓ . Starting from this point, we shall for convenience work exclusively over \mathbf{C} ; also, since \mathcal{R}_1 and \mathcal{R}_2 are the full rings of modular forms on $\Gamma(1)$ and $\Gamma(2)$, we can restrict to $\ell \geq 3$. We use the standard notation for the spaces of cusp forms and modular forms of weight j on a congruence subgroup Γ :

$$(4.22) \quad \mathcal{S}_j(\Gamma) = \{\text{cusp forms}\} \subset \mathcal{M}_j(\Gamma) = \{\text{holomorphic modular forms over } \mathbf{C}\}.$$

We also make use of the usual group action $f \mapsto f|_j \gamma$ of $\gamma \in \Gamma(1)$ on the space $\mathcal{M}_j(\Gamma(\ell))$. This action of course preserves the spaces of Eisenstein series and cusp forms in all weights. We interchangeably view γ as an element of $\Gamma(1)$ or of $\Gamma(1)/\Gamma(\ell) \cong SL(2, \mathbf{Z}/\ell\mathbf{Z})$. This group also acts on the torsion group $E[\ell]$ while preserving the Weil pairing, and on our ring \mathcal{R}_ℓ via ring isomorphisms. Indeed, for

$\gamma \in SL(2, \mathbf{Z}/\ell\mathbf{Z})$, we have $P \mapsto P \cdot \gamma$, where

$$(4.23) \quad \begin{aligned} z_P = \frac{a_1\tau + a_2}{\ell} &\implies z_{P \cdot \gamma} = \frac{a_1'\tau + a_2'}{\ell} \text{ with } (a_1' \ a_2') = (a_1 \ a_2)\gamma, \\ P \in E[\ell] &\implies \lambda_P|_1\gamma = \lambda_{P \cdot \gamma}. \end{aligned}$$

We briefly review the well-known interpretation of Hecke operators in terms of a trace between congruence subgroups. Given a Hecke operator described as a double coset $\Gamma(\ell)\alpha\Gamma(\ell)$ with $\alpha \in GL^+(2, \mathbf{Q})$, we can harmlessly multiply α by a scalar to obtain a primitive integral matrix; then composing this double coset on the left and right by the action of elements $\gamma_1, \gamma_2 \in \Gamma(1)$ allows us to assume without loss of generality that $\alpha = \begin{pmatrix} n & \\ & 1 \end{pmatrix}$ for some $n \geq 1$. We then have, for $f(\tau) \in \mathcal{M}_j(\Gamma(\ell))$:

$$(4.24) \quad f|_j\Gamma(\ell) \begin{pmatrix} n & \\ & 1 \end{pmatrix} \Gamma(\ell) = C \sum_{\gamma \in \Gamma(n\ell) \backslash \Gamma(\ell)} (f(n\tau))|_j\gamma,$$

where $C = C_{n,\ell,j}$ is a suitable normalizing constant. Note that if $f(\tau) \in \mathcal{R}_\ell$, then $f(n\tau) \in \mathcal{R}_{n\ell}$; indeed, the map $f \mapsto f(n\tau)$ respects multiplication of forms, so it is enough to check the above statement for the weight 1 Eisenstein series $\lambda_P = -G_1(\tau, P)$ that generate \mathcal{R}_ℓ . This is just the identity

$$(4.25) \quad G_1(n\tau, \frac{a_1\tau + a_2}{\ell}) = n^{-1} \sum_{k \bmod n} G_1(\tau, \frac{a_1n\tau + a_2 + k\ell}{n\ell}).$$

The sum over representatives $\gamma \in \Gamma(n\ell) \backslash \Gamma(\ell)$ in (4.24) is a trace from $\mathcal{M}_j(\Gamma(n\ell))$ to $\mathcal{M}_j(\Gamma(\ell))$, and we shall henceforth work with it instead of with double cosets.

With these preliminaries out of the way, we can state and prove our result for weight 2, which will be superseded later when we show that \mathcal{R}_ℓ contains all of $\mathcal{M}_2(\Gamma(\ell))$.

Proposition 4.8. *Let $k = \mathbf{C}$. Then the trace of a weight 2 element of $\mathcal{R}_{n\ell}$ from $\mathcal{M}_2(\Gamma(n\ell))$ to $\mathcal{M}_2(\Gamma(\ell))$ actually belongs to \mathcal{R}_ℓ . (A priori, this trace merely belongs to $\mathcal{R}_{n\ell} \cap \mathcal{M}_2(\Gamma(\ell))$.)*

Corollary 4.9. *Over \mathbf{C} , the weight 2 part of \mathcal{R}_ℓ is stable under the action of the Hecke algebra for $\Gamma(\ell)$.*

Proof of Proposition 4.8. By the observation immediately following Remark 4.7, we can assume that $\ell \geq 3$; it is enough to show in that case that the trace of any product $\lambda_P\lambda_Q = G_1(\tau, P)G_1(\tau, Q)$ with $P, Q \in E[n\ell] - \{P_0\}$ belongs to \mathcal{R}_ℓ . Now \mathcal{R}_ℓ already contains all the Eisenstein series on $\Gamma(\ell)$ in weight 2 (indeed, in all weights j , by Theorem 3.9), so we can work modulo Eisenstein series in our proof. As we mentioned in Remark 4.7, this can be done even if we encounter Eisenstein series of higher level in some intermediate steps. Furthermore, the trace down from level $n\ell$ to level ℓ can be done one prime factor at a time, so we may harmlessly assume that n is a prime number. There are two cases to consider: (i) n is prime and $n \nmid \ell$, and (ii) n is prime and $n|\ell$.

In case (i), we take a direct sum decomposition $E[n\ell] = E[\ell] \oplus E[n]$, and note that $\Gamma(n\ell) \backslash \Gamma(\ell)$ is isomorphic to $SL(2, \mathbf{Z}/n\mathbf{Z})$ and that it affects only the $E[n]$ part. Decompose $P = A \oplus T_0$ and $Q = B \oplus U_0$, with $A, B \in E[\ell]$ and $T_0, U_0 \in E[n]$. We may suppose that one of $\{T_0, U_0\}$ — let us say, T_0 — is not equal to P_0 , since otherwise $\lambda_P\lambda_Q \in \mathcal{R}_\ell$ already. Then there are two subcases: (i.a) there exists $s \in \mathbf{Z}/n\mathbf{Z}$ ($s = 0$ is allowed) such that $U_0 = [s]T_0$, and (i.b) $\{T_0, U_0\}$ are a basis

for $E[n]$. In subcase (i.a), the trace of $\lambda_{A \oplus T_0} \lambda_{B \oplus [s]T_0}$ is equal to a multiple of $\sum_{T \in E[n] - \{P_0\}} \lambda_{A \oplus T} \lambda_{B \oplus [s]T}$. By Proposition 4.6, this is congruent modulo \mathcal{E}_2 to an element of \mathcal{R}_ℓ (the “missing term” in the sum, corresponding to $T = P_0$, is $\lambda_A \lambda_B$, which already belongs to \mathcal{R}_ℓ). Hence the trace itself belongs to \mathcal{R}_ℓ , as we have observed before.

In subcase (i.b), let $\zeta = e_n(T_0, U_0)$, which is a primitive n th root of unity. Then the trace that we wish to compute is

$$(4.26) \quad \sum_{\substack{T, U \in E[n] \\ e_n(T, U) = \zeta}} \lambda_{A \oplus T} \lambda_{B \oplus U} = \frac{-1}{n\ell} \sum_{\substack{T, U \in E[n] \\ e_n(T, U) = \zeta \\ V \in E[n], C \in E[\ell]}} \lambda_{A \oplus T} \lambda_{C \oplus V} e_{n\ell}(C \oplus V, B \oplus U).$$

Here we have invoked (4.3), where we let $C \oplus V$ range over the elements of $E[n\ell]$. Now $e_{n\ell}(C \oplus V, B \oplus U) = e_\ell([n]C, B) e_n([\ell]V, U)$, so the quantity in (4.26) is a linear combination of terms (indexed by C) of the form

$$(4.27) \quad \sum_{\substack{T, U, V \in E[n] \\ e_n(T, U) = \zeta}} \lambda_{A \oplus T} \lambda_{C \oplus V} e_n([\ell]V, U).$$

For fixed T and V , we must hence study the sum over those U for which $e_n(T, U) = \zeta$. Such a U exists if and only if $T \neq P_0$ (here we use the facts that n is prime and $\zeta \neq 1$), in which case U ranges over the set of torsion points $\{U_T \oplus [t]T \mid t \in \mathbf{Z}/n\mathbf{Z}\}$ for some particular choice of U_T (depending on T) with $e_n(T, U_T) = \zeta$. The sum over U thus contains a factor $\sum_{t \in \mathbf{Z}/n\mathbf{Z}} e_n([\ell]V, U_T \oplus [t]T)$, which vanishes unless V belongs to the cyclic subgroup generated by T (recall that ℓ is relatively prime to n). We obtain that (4.27) is equal to

$$(4.28) \quad \begin{aligned} & \sum_{T \in E[n] - \{P_0\}} \sum_{V \text{ of the form } V = [s]T} \lambda_{A \oplus T} \lambda_{C \oplus [s]T} \cdot n e_n([\ell][s]T, U_T) \\ &= \sum_{s \in \mathbf{Z}/n\mathbf{Z}} n \zeta^{\ell s} \sum_{T \neq P_0} \lambda_{A \oplus T} \lambda_{C \oplus [s]T}, \end{aligned}$$

which brings us back to subcase (i.a).

We now turn to case (ii). We write $\ell = Ln^k$ with $n \nmid L$ and $k \geq 1$, and decompose $P = A \oplus T_0$ and $Q = B \oplus U_0$ with $A, B \in E[L]$ and $T_0, U_0 \in E[n^k]$. We wish to compute a trace using representatives for $\Gamma(Ln^{k+1}) \backslash \Gamma(Ln^k)$. Such representatives again do not affect A or B , and their action on T_0 and U_0 can be described by matrices in $SL(2, \mathbf{Z}/n^{k+1}\mathbf{Z})$ that are congruent to the identity modulo n^k ; thus such matrices have the form

$$(4.29) \quad \begin{pmatrix} 1 + n^k \alpha & n^k \beta \\ n^k \gamma & 1 - n^k \alpha \end{pmatrix} = I + n^k M, \quad M = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \in M_2^{\text{trace } 0}(\mathbf{Z}/n\mathbf{Z}).$$

The reader should note that we view the entries of M as being in $\mathbf{Z}/n\mathbf{Z}$, but that multiplying them by n^k yields elements of $n^k \mathbf{Z}/n^{k+1}\mathbf{Z}$, not zero. We also point out that we shall feel free to use other bases for $E[n^{k+1}] \cong (\mathbf{Z}/n^{k+1}\mathbf{Z})^2$ than the standard basis $\{P_{\tau/n^{k+1}}, P_{1/n^{k+1}}\}$; even if the change of basis does not have determinant 1 (and hence changes the Weil pairing), our description of M in (4.29) remains valid. Let us write $\hat{T}_0 = [n^k]T_0$ and $\hat{U}_0 = [n^k]U_0$. We have $\hat{T}_0, \hat{U}_0 \in E[n]$,

and the trace that we wish to compute is then

$$(4.30) \quad \sum_{M \in M_2^{\text{trace } 0}(\mathbf{Z}/n\mathbf{Z})} \lambda_{A \oplus T_0 \oplus (\hat{T}_0 \cdot M)} \lambda_{B \oplus U_0 \oplus (\hat{U}_0 \cdot M)},$$

where the action of M is analogous to that in (4.23). Once again we may suppose that \hat{T}_0 and \hat{U}_0 are not both P_0 (otherwise, $T_0, U_0 \in E[n^k]$ and we are already in $\mathcal{R}_{n^k L} = \mathcal{R}_\ell$), and that without loss of generality $\hat{T}_0 \neq P_0$. We face analogous subcases: (ii.a) there exists $s \in \mathbf{Z}/n\mathbf{Z}$ such that $\hat{U}_0 = [s]\hat{T}_0$ and (ii.b) $\{\hat{T}_0, \hat{U}_0\}$ are a basis for $E[n]$.

In subcase (ii.a), the points $\hat{T}_0 \cdot M$ cover all of $E[n]$ (including P_0), each point $\hat{T} \in E[n]$ occurring n times. (The easiest way to see this is to write M with respect to a basis for $E[n]$ that includes \hat{T}_0 .) Hence we obtain that (4.30) is a multiple of

$$(4.31) \quad \sum_{\hat{T} \in E[n]} \lambda_{A \oplus T_0 \oplus \hat{T}} \lambda_{B \oplus U_0 \oplus [s]\hat{T}}.$$

Modulo Eisenstein series, this last expression is a linear combination of terms of the form

$$(4.32) \quad \lambda_{[a](A \oplus T_0) \oplus [b](B \oplus U_0)} \lambda_{[c](A \oplus T_0) \oplus [d](B \oplus U_0)}, \text{ for } a + sb \equiv c + sd \equiv 0 \pmod{n}.$$

We observe that $[n^k]([a]T_0 \oplus [b]U_0) = [a]\hat{T}_0 \oplus [b]\hat{U}_0 = [a + sb]\hat{T}_0 = P_0$, whereas $[a]A \oplus [b]B \in E[L]$, so the first factor in (4.32) involves torsion points in $E[n^k L] = E[\ell]$; an analogous statement holds for the second factor, and we obtain an element of \mathcal{R}_ℓ , as desired.

In subcase (ii.b), we write M in terms of the basis $\{\hat{T}_0, \hat{U}_0\}$, and obtain that we wish to study

$$(4.33) \quad \sum_{\alpha, \beta, \gamma \in \mathbf{Z}/n\mathbf{Z}} \lambda_{A \oplus T_0 \oplus [\alpha]\hat{T}_0 \oplus [\beta]\hat{U}_0} \lambda_{B \oplus U_0 \oplus [\gamma]\hat{T}_0 \oplus [-\alpha]\hat{U}_0}.$$

Similarly to subcase (i.b), we rewrite the factor $\lambda_{B \oplus U_0 \oplus [\gamma]\hat{T}_0 \oplus [-\alpha]\hat{U}_0}$ in terms of the Weil pairing and $\lambda_{C \oplus V}$, for $C \in E[L]$ and $V \in E[n^{k+1}]$. We obtain a linear combination of terms of the following form (here the triples $(\alpha, \beta, \gamma) \in (\mathbf{Z}/n\mathbf{Z})^3$ are analogous to the pairs $\{(T, U) \in E[n] \times E[n] \mid e_n(T, U) = \zeta\}$ of (4.27)):

$$(4.34) \quad \sum_{\substack{\alpha, \beta, \gamma \in \mathbf{Z}/n\mathbf{Z} \\ V \in E[n^{k+1}]}} \lambda_{A \oplus T_0 \oplus [\alpha]\hat{T}_0 \oplus [\beta]\hat{U}_0} \lambda_{C \oplus V} e_{n^{k+1}}([L]V, U_0 \oplus [\gamma]\hat{T}_0 \oplus [-\alpha]\hat{U}_0).$$

Now perform the sum over γ first: the inner factor $\sum_\gamma e_{n^{k+1}}([L]V, [\gamma]\hat{T}_0)$ can be rewritten as $\sum_\gamma e_n([L]([n^k]V), [\gamma]\hat{T}_0)$ with $[n^k]V \in E[n]$. Thus, as in case (i.b), the only terms that survive are those where $[n^k]V = [s]\hat{T}_0 = [sn^k]T_0$ for some $s \in \mathbf{Z}/n\mathbf{Z}$. Equivalently, we can write $V = [s]T_0 \oplus W$ for some s and for some $W \in E[n^k]$. In such a situation, we have $e_{n^{k+1}}([L]V, [-\alpha]\hat{U}_0) = e_n([n^k L]V, [-\alpha]\hat{U}_0) = e_n([Ls]\hat{T}_0, [-\alpha]\hat{U}_0) = e_n([-Ls]([\alpha]\hat{T}_0 \oplus [\beta]\hat{U}_0), \hat{U}_0)$. At this point, we note that as α and β vary, the point $\hat{T} := [\alpha]\hat{T}_0 \oplus [\beta]\hat{U}_0$ runs over all points of $E[n]$. Putting all this together, we obtain that our expression is a linear combination of terms

(indexed by C and s) of the form

$$(4.35) \quad \sum_{\substack{\hat{T} \in E[n] \\ W \in E[n^k]}} \lambda_{A \oplus T_0 \oplus \hat{T}} \lambda_{C \oplus [s]T_0 \oplus W} e_{n^{k+1}}([Ls]T_0 \oplus [L]W, U_0) e_n([-Ls]\hat{T}, \hat{U}_0).$$

The above expression contains a common factor $e_{n^{k+1}}([Ls]T_0, U_0)$. We can also rewrite

$$(4.36) \quad e_{n^{k+1}}([L]W, U_0) e_n([-Ls]\hat{T}, \hat{U}_0) = e_{n^{k+1}}([L](W \ominus [s]\hat{T}), U_0).$$

We define $X := W \ominus [s]\hat{T} \in E[n^k]$; this yields a bijection from the set $E[n] \times E[n^k]$ to itself, sending the pair (\hat{T}, W) to the pair (\hat{T}, X) . Our expression (4.35) then becomes

$$(4.37) \quad e_{n^{k+1}}([Ls]T_0, U_0) \sum_{\substack{\hat{T} \in E[n] \\ X \in E[n^k]}} \lambda_{A \oplus T_0 \oplus \hat{T}} \lambda_{C \oplus [s]T_0 \oplus X \oplus [s]\hat{T}} e_{n^{k+1}}([L]X, U_0).$$

Writing the sum in the order $\sum_X \sum_{\hat{T}}$, we see that the inner sum over \hat{T} is now exactly analogous to (4.31). In our setting, C and $[s]T_0 \oplus X$ play the roles of B and U_0 from (4.31)³, and we obtain by an identical argument to the subcase (ii.a) that our final expression is congruent modulo Eisenstein series to an element of \mathcal{R}_ℓ . This completes the proof. \square

Having disposed of weight 2, we now turn our attention to weight 3. We shall prove weight 3 analogs of Propositions 4.6 and 4.8, but only for modular forms of the form $x_P \lambda_Q$, i.e., for products of an Eisenstein series of weight 2 with an Eisenstein series of weight 1. We shall continue to work modulo Eisenstein series, i.e., modulo the space \mathcal{E}_3 . In this context, the analog of (4.9) is the following statement, which holds whenever $P \oplus Q \oplus R = P_0$:

$$(4.38) \quad (x_P - x_R)(\lambda_P + \lambda_Q + \lambda_R) \in \mathcal{E}_3.$$

To see this, first observe by Proposition 3.5 that if none of P , Q , and R is equal to P_0 , then the above expression is equal to $y_P - y_R$, which is in \mathcal{E}_3 . On the other hand, if one of the points is P_0 , then $\lambda_P + \lambda_Q + \lambda_R = 0$ by our conventions, and the above expression is equal to 0.

The next lemma is the weight 3 analog of the key computational step that we did in (4.19). We note incidentally that we could have applied the techniques of this lemma to the weight 2 identity $(\lambda_P + \lambda_Q + \lambda_R)^2 = x_P + x_Q + x_R$. This would have yielded a slightly weaker result than Proposition 4.6 (analogous to the proof below in weight 3) that would also have been sufficient for our purposes.

Lemma 4.10. *Let $k = \mathbf{C}$, let $n \geq 1$, and let $A, B \in E[n\ell]$ (typically with $A, B \in E[\ell]$). Then we have the following congruences modulo \mathcal{E}_3 :*

$$(4.39) \quad \sum_{T \in E[n]} x_{A \oplus T} \lambda_{A \oplus T} \equiv nx_{[n]A} \lambda_{[n]A},$$

³ Recall that U_0 in (4.31) had the property that $[n^k]U_0 = \hat{U}_0 = [s]\hat{T}_0$. The analogous observation in our setting is that $[n^k]([s]T_0 \oplus X) = [s]\hat{T}_0$.

$$\begin{aligned}
(4.40) \quad \sum_{T \in E[n]} x_{A \oplus T} \lambda_{B \oplus T} &\equiv -n x_{[n]A} \lambda_{[n]A} + n^2 x_{[n]A} \lambda_{A \oplus B} \\
&\quad + n x_{A \oplus B} \lambda_{[n]A} + n x_{A \oplus B} \lambda_{[n]B} - n^2 x_{A \oplus B} \lambda_{A \oplus B}.
\end{aligned}$$

Proof. To show (4.39), we take $P = A \oplus T$, $Q = \ominus A \oplus U$, and $R = \ominus T \oplus U$ in (4.38), and we sum the result over all $T, U \in E[n]$, knowing that the final result will be $\equiv 0$ modulo \mathcal{E}_3 . We now observe that

$$\begin{aligned}
(4.41) \quad \sum_{T, U \in E[n]} x_{A \oplus T} \lambda_{A \oplus T} &= n^2 \sum_T x_{A \oplus T} \lambda_{A \oplus T}, \\
\sum_{T, U} x_{A \oplus T} \lambda_{\ominus A \oplus U} &= n^2 x_{[n]A} \cdot n \lambda_{[-n]A} = -n^3 x_{[n]A} \lambda_{[n]A}, \\
\sum_{T, U} x_{A \oplus T} \lambda_{\ominus T \oplus U} &= \sum_{T, V \in E[n]} x_{A \oplus T} \lambda_V = 0, \\
\sum_{T, U} x_{\ominus T \oplus U} \lambda_{A \oplus T} &= \sum_{T, V} x_V \lambda_{A \oplus T} = 0; \text{ similarly, } \sum_{T, U} x_{\ominus T \oplus U} \lambda_{\ominus A \oplus U} = 0, \\
\sum_{T, U} x_{\ominus T \oplus U} \lambda_{\ominus T \oplus U} &= n^2 \sum_V x_V \lambda_V = n^2 \sum_V x_{\ominus V} \lambda_{\ominus V} = -(\text{itself}) = 0,
\end{aligned}$$

where we have used (4.10) and (3.16) as needed.

For the proof of (4.40), we take the sum over all T in $E[n]$ of (4.38) with $P = A \oplus T$, $Q = B \oplus T$, and $R = \ominus A \oplus B$ (so $\lambda_R = -\lambda_{A \oplus B}$ and $x_R = x_{A \oplus B}$). We then proceed as in the proof of (4.39), while using (4.39) at one point, to obtain the desired result. \square

At this point, the generalization of Propositions 4.6 and 4.8 to weight 3 is straightforward.

Proposition 4.11. *Make the same hypotheses as Lemma 4.10, and let $s \in \mathbf{Z}$. Then we have the following congruence modulo \mathcal{E}_3 :*

$$\begin{aligned}
(4.42) \quad \sum_{T \in E[n]} x_{A \oplus T} \lambda_{B \oplus [s]T} \\
\equiv (a \text{ linear combination of terms of the form } x_{[a]A \oplus [b]B} \lambda_{[c]A \oplus [d]B}), \\
\text{with } a - sb \equiv c - sd \equiv 0 \pmod{n}.
\end{aligned}$$

(An analogous statement holds for sums $\sum_T x_{A \oplus [s]T} \lambda_{B \oplus T}$, in which case the congruence condition modulo n becomes $-sa + b \equiv -sc + d \equiv 0$.) We remark incidentally that the determinant of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ above need not be equal to n .

Furthermore, if $P, Q \in E[n\ell]$, then the trace of the weight 3 element $x_P \lambda_Q \in \mathcal{R}_{n\ell}$ down to level $\Gamma(\ell)$ is congruent modulo \mathcal{E}_3 to a linear combination of terms $x_R \lambda_S \in \mathcal{R}_\ell$, with $R, S \in E[\ell]$.

Proof. The proof of (4.42) follows the same lines as the proof of Proposition 4.6, with the same type of induction on s . For $s = 0$, it follows as usual from (4.10), and we have already proved the case $s = 1$ in (4.40). The key step in the induction (analogous to (4.19)) amounts to applying (4.40) to the T -part of the sum $\sum_{T, U} x_{A \oplus T} \lambda_{B' \oplus T \oplus U}$. The ideas are essentially the same as before, with the use of (4.39) thrown in for good measure. (It is worth pointing out that while carrying out the same proof in the case of $\sum_T x_{A \oplus [s]T} \lambda_{B \oplus T}$, we encounter at one stage

the sum $\sum_{U \in E[s]} x_{[n]A' \oplus [n]U} \lambda_{[n]A' \oplus [n]U}$, where $[s]A' = A$. Write $d = \gcd(n, s)$ and $\hat{s} = s/d$; then our sum becomes $d^2 \cdot \sum_{\hat{U} \in E[\hat{s}]} x_{[n]A' \oplus \hat{U}} \lambda_{[n]A' \oplus \hat{U}}$, which we simplify using (4.39).)

As for the proof of the statement about the trace of $x_P \lambda_Q$, it follows the argument of Proposition 4.8 with only trivial changes. The only point worth mentioning is that the roles of T_0 and U_0 are no longer symmetric, so we cannot simply assume that T_0 in case (i) (respectively, \hat{T}_0 in case (ii)) is not equal to P_0 . However, if T_0 (respectively, \hat{T}_0) is equal to P_0 , then $P \in E[\ell]$ already, and the trace is then equal to $x_P \operatorname{tr}(\lambda_Q)$, which is easy to analyze using (4.10), or for that matter by noting that the trace of the Eisenstein series λ_Q is again an Eisenstein series. \square

5. GENERATING ALL MODULAR FORMS IN WEIGHTS ≥ 2 , AND A MODEL FOR $X(\ell)$

We are now ready to use Propositions 4.8 and 4.11 to show that the ring of modular forms over \mathbf{C} generated by the Eisenstein series of weight 1 contains all modular forms in weights 2 and above. This is Theorem 5.1 below. We then apply the result to obtain a convenient method to find explicit models for the modular curve $X(\ell)$, in Theorem 5.5 below.

We prove Theorem 5.1 via relating the result to the nonvanishing of a special value of an L -function, which is also the strategy of [BG01b, BG03]. Our proof brings in the L -function via a Rankin-Selberg integral, in contrast to the calculations in the articles by Borisov and Gunnells, which use q -expansions whose coefficients are modular symbols. It is worth noting that one can give a much simpler proof of the (rather weaker) fact that \mathcal{R}_ℓ contains all modular forms in sufficiently high weights. To see this, note that the ring of all modular forms is the graded integral closure of \mathcal{R}_ℓ in its own field of fractions, by Definition 3.2 and Remark 3.3. Hence $X(\ell) = \mathbf{Proj} \mathcal{R}_\ell$; since $X(\ell)$ is nonsingular, it is then a standard fact that the graded components of the two rings (\mathcal{R}_ℓ and the ring of modular forms) agree in sufficiently high weights — see for example [Har77], Section II.5.19 and Exercises II.5.9, II.5.14. Precise but large bounds for the meaning of “sufficiently high” for arbitrary curves are given in [GLP83], but they of course grow with the genus of the curve, which for $X(\ell)$ is $O(\ell^3)$. The interest of our results, as well as those of Borisov-Gunnells, is that they give a fixed value for “sufficiently high”: 2 in our result for $\Gamma(\ell)$, and 3 for their result for $\Gamma_1(\ell)$ (where they obtain all cusp forms modulo Eisenstein series, but potentially miss some Eisenstein series).

Theorem 5.1. *Let $k = \mathbf{C}$. Then \mathcal{R}_ℓ contains all modular forms on $\Gamma(\ell)$ of weight 2 and above. In other words, \mathcal{R}_ℓ “misses” precisely the cusp forms in weight 1.*

Proof. Since \mathcal{R}_ℓ contains all modular forms for $\ell \leq 2$, we as usual restrict to the case $\ell \geq 3$. Our first claim is that it is enough to show that \mathcal{R}_ℓ contains all of $\mathcal{M}_2(\Gamma(\ell))$ and $\mathcal{M}_3(\Gamma(\ell))$. To see this claim, observe that $\Gamma(\ell)$ has no elliptic elements or irregular cusps; hence there exists a line bundle \mathcal{L} on $X(\ell)$ such that for all weights j , we have $\mathcal{M}_j(\Gamma(\ell)) = H^0(X(\ell), \mathcal{L}^{\otimes j})$. Moreover, elements of \mathcal{M}_2 can be viewed as 1-forms on $X(\ell)$ with at worst a simple pole at each cusp. Hence the degree of $\mathcal{L}^{\otimes 2}$ is equal to $2g - 2 + \kappa$, where g is the genus of $X(\ell)$, and κ is the number of cusps. Since $\kappa \geq 4$ for $\ell \geq 3$, by standard formulas for modular curves (e.g., Section 1.6 of [Shi71]), we obtain that $2 \deg \mathcal{L} \geq 2g + 2$. This is enough to show that the multiplication map $\mathcal{M}_j(\Gamma(\ell)) \otimes \mathcal{M}_{j'}(\Gamma(\ell)) \rightarrow \mathcal{M}_{j+j'}(\Gamma(\ell))$ is surjective for $j, j' \geq 2$, since the degrees of $\mathcal{L}^{\otimes j}$ and $\mathcal{L}^{\otimes j'}$ are both $\geq 2g + 1$ (for a sketch of this

standard result, see Lemma 2.2 of [KM04]; the survey in Section 1 of [Laz89] is also a particularly useful reference). This implies that any ring of modular forms that contains $\mathcal{M}_2(\Gamma(\ell))$ and $\mathcal{M}_3(\Gamma(\ell))$ must contain all forms in higher weights, thereby establishing our claim.

We therefore turn to the situation in weights 2 and 3, which we study using a result of Shimura [Shi76]. This result states that a suitable Rankin-Selberg convolution of a newform F with an Eisenstein series gives a product of two special values of Hecke L -functions associated to F and to Dirichlet characters ξ, ψ . More precisely, Theorem 2 (with $r = 0$) of [Shi76], and equation (4.3) of the same article (with $k = j \geq 2$, $l = 1$, and $m = j - 1$) imply the following statement, which holds for any $j \geq 2$: let $F \in \mathcal{S}_j$ be a newform with character χ , and let ξ, ψ be Dirichlet characters with $(\xi\psi)(-1) = -1$; then there exists a product GG' of two Eisenstein series, with $G \in \mathcal{E}_1$ and $G' \in \mathcal{E}_{j-1}$, such that the Petersson inner product of F with GG' gives

$$(5.1) \quad \langle F, GG' \rangle = C \cdot L(j-1, F, \xi) L(j-1, F, \psi)$$

with an explicit nonzero constant C . (Here, if $j = 3$, we must have $\chi\xi\psi \neq 1$ in order for $G' \in \mathcal{E}_2$ to be holomorphic.) Note that we have normalized the Petersson inner product so that it is insensitive to the choice of common congruence subgroup Γ with respect to which F , G , and G' are all invariant.

We deduce from (5.1) that for a given F , we can choose ξ and ψ (and, with them, G and G') so as to make the above inner product nonzero. Indeed, when $j \geq 3$, then, regardless of ξ and ψ , the L -functions on the right side are nonzero, since they are evaluated outside the critical strip if $j \geq 4$, and at the edge of the critical strip if $j = 3$ (see, e.g., Proposition 2 of [Shi76], or [JS77] for a more general result). Thus we can also ensure that $\chi\xi\psi \neq 1$ as needed in the special case $j = 3$. On the other hand, if $j = 2$, then, by Theorem 2 of [Shi77], there exist ξ and ψ for which the right side of (5.1) is nonzero.

We can now show that \mathcal{R}_ℓ contains all of $\mathcal{M}_2(\Gamma(\ell))$ and $\mathcal{M}_3(\Gamma(\ell))$. Since \mathcal{R}_ℓ contains all Eisenstein series on $\Gamma(\ell)$, we are reduced to checking whether \mathcal{R}_ℓ contains all of $\mathcal{S}_j(\Gamma(\ell))$ for $j \in \{2, 3\}$, or alternatively to checking that the orthogonal complement $[\mathcal{R}_\ell \cap \mathcal{S}_j(\Gamma(\ell))]^\perp$ in $\mathcal{S}_j(\Gamma(\ell))$ is zero. Let $0 \neq f \in \mathcal{S}_j(\Gamma(\ell))$ be any nonzero cuspform in this orthogonal complement. Then there exist constants $c_1, \dots, c_N \in \mathbf{C}$ and matrices $\alpha_1, \dots, \alpha_N \in GL^+(2, \mathbf{Q})$ such that the linear combination $F = \sum_i c_i f|_{\alpha_i}$ is actually a newform (for instance, use an element of the Hecke algebra to project to a single automorphic representation, and then move around within it to reach the newform). We can find G, G' as above such that $\langle F, GG' \rangle \neq 0$. But this means that

$$(5.2) \quad 0 \neq \langle \sum_i c_i f|_{\alpha_i}, GG' \rangle = \sum_i c_i \langle f, (GG')|_{\alpha_i^{-1}} \rangle.$$

In the above expression, each form $(GG')|_{\alpha_i^{-1}} = (G|_{\alpha_i^{-1}})(G'|_{\alpha_i^{-1}})$ is still the product of an Eisenstein series of weight 1 with an Eisenstein series of weight $j-1 \in \{1, 2\}$; hence it can be written as a linear combination of modular forms of the form $\lambda_P \lambda_Q$ or $\lambda_P x_Q$, with $P, Q \in E[n\ell]$ for some (possibly rather large) n . We obtain a linear combination of inner products of the form $\langle f, \lambda_P \lambda_Q \rangle$ or $\langle f, \lambda_P x_Q \rangle$, which can in turn be reexpressed (up to a constant factor) as an inner product of the form $\langle f, \text{tr}_{\Gamma(\ell)}^{\Gamma(n\ell)} \lambda_P \lambda_Q \rangle$ or $\langle f, \text{tr}_{\Gamma(\ell)}^{\Gamma(n\ell)} \lambda_P x_Q \rangle$, and the traces belong to \mathcal{R}_ℓ by Propositions 4.8 and 4.11. Furthermore, the Eisenstein part of each such trace, and

therefore also the cuspidal part, must then belong to \mathcal{R}_ℓ . Thus the inner products must all be zero if f belongs to the orthogonal complement $[\mathcal{R}_\ell \cap \mathcal{S}_j(\Gamma(\ell))]^\perp$ in question. This contradicts the fact that $\langle F, GG' \rangle \neq 0$, and we deduce that the orthogonal complement is zero after all. This concludes our proof. \square

Theorem 5.1 makes it possible to compute nice models for the modular curve $X(\ell)$. These models are defined over $\mathbf{Q}(\mu_\ell)$ (we suspect that a more careful investigation would yield models over \mathbf{Q}), and are in the form called “Representation B” in [KM07]. The basic idea is to work implicitly with the projective embedding of X given by a line bundle $\hat{\mathcal{L}}$, with $\deg \hat{\mathcal{L}} \geq 2g + 2$, for which it is a standard fact that the ideal of equations defining the image of X is generated by quadrics (see, for example, Section 1 of [Laz89]). (In our setting, we will have $X = X(\ell)$ and $\hat{\mathcal{L}} = \mathcal{L}^{\otimes 2}$, in the notation of the proof of Theorem 5.1.) We then define $V = H^0(X, \hat{\mathcal{L}})$ and $V' = H^0(X, \hat{\mathcal{L}}^{\otimes 2})$; in our setting, this means that $V = \mathcal{M}_2(\Gamma(\ell))$ and $V' = \mathcal{M}_4(\Gamma(\ell))$. Let μ be the multiplication map $\mu : V \otimes V \rightarrow V'$, and note that μ factors through a map $\bar{\mu} : \text{Sym}^2 V \rightarrow V'$. Then the kernel of $\bar{\mu}$ describes exactly the quadric equations that define X in its projective embedding, and hence X can be recovered from a knowledge of the spaces V and V' and of the multiplication map μ . Now it is possible to represent μ by a multiplication table in terms of bases for V and V' (this was called “Representation A” in [KM07]), but a superior method is to take a collection of points p_1, \dots, p_N of points on X , with $N > 2 \deg \hat{\mathcal{L}}$, and to represent elements of V and V' by their “values” at the points p_i ; this presupposes some fixed choice of trivialization of $\hat{\mathcal{L}}$ in a neighborhood of each p_i . It turns out that the points p_i need not be distinct, provided we replace the value of an element $s \in V$ (or V') by its n th order Taylor expansion at a point that appears with multiplicity n . A better point of view is to replace the points p_i by the effective divisor $D = (p_1) + \dots + (p_N)$ on X , and reformulating the value of $s \in V = H^0(X, \hat{\mathcal{L}})$ at the points of D in terms of the image of s in $H^0(X, \hat{\mathcal{L}})/H^0(X, \hat{\mathcal{L}}(-D)) \cong H^0(X, \hat{\mathcal{L}}/\hat{\mathcal{L}}(-D))$. The local trivialization of $\hat{\mathcal{L}}$ then amounts to fixing once and for all an isomorphism between the sheaves of \mathcal{O}_X -modules $\hat{\mathcal{L}}/\hat{\mathcal{L}}(-D)$ and $\mathcal{O}_X/\mathcal{O}_X(-D) = \mathcal{O}_D$, which are supported on the possibly nonreduced zero-dimensional subscheme D of X . Thus our “values” in $H^0(X, \hat{\mathcal{L}}/\hat{\mathcal{L}}(-D))$ are interpreted as elements of the finite-dimensional algebra $\mathcal{A} = H^0(X, \mathcal{O}_D)$. A similar assertion works for the values of an element in V' (using the induced isomorphism between $\hat{\mathcal{L}}^{\otimes 2}/\hat{\mathcal{L}}^{\otimes 2}(-D)$ and \mathcal{O}_D), and in this case the multiplication map μ amounts to the multiplication in \mathcal{A} . All our ingredients are now in place to give the definition of Representation B.

Definition 5.2. Let X be a smooth projective curve over a base field F , and choose a line bundle $\hat{\mathcal{L}}$ and an effective divisor D on X that are both F -rational. Assume moreover that $\deg \hat{\mathcal{L}} \geq 2g + 2$ and that $\deg D > 2 \deg \hat{\mathcal{L}}$, as discussed above. Then Representation B of the curve X is given by the finite-dimensional F -algebra $\mathcal{A} = H^0(X, \mathcal{O}_D)$, along with F -subspaces $V, V' \subset \mathcal{A}$. Here we have replaced V by its image under the F -linear map $H^0(X, \hat{\mathcal{L}}) \rightarrow H^0(X, \hat{\mathcal{L}}/\hat{\mathcal{L}}(-D)) \cong \mathcal{A}$, and similarly for V' with respect to $\hat{\mathcal{L}}^{\otimes 2}$. (The condition on $\deg D$ ensures that these two F -linear maps are injections.) The multiplication map μ is simply the restriction to V of the multiplication in \mathcal{A} , and this yields sufficient information to deduce the set of quadric equations that define the image of X in projective space defined by the embedding associated to $\hat{\mathcal{L}}$.

We point out that the precise definition of Representation B in [KM07] also specifies that \mathcal{A} is represented as a product of rings that are explicitly given in the form $F[x]/(f(x))$; this shall not concern us here.

Remark 5.3. In our setting, where V and V' are spaces of modular forms, we can take D to be a multiple of the cusp at infinity; then the values at D are q -expansions of modular forms up to $O(q^N)$, and these q -expansions give rise to equations for modular curves via the approach sketched above. This approach has already appeared in the literature; see [Gal96] and Section 2 of [BGJGP05], where $\hat{\mathcal{L}}$ is replaced by the canonical bundle, and the projective embedding is then replaced with the canonical embedding (if the curve is not hyperelliptic), with some modifications since the ideal of a canonical curve occasionally requires generators going up to degree 4.

One novel aspect of our approach is that we evaluate the modular forms at noncuspidal points; we hope that this approach, suitably developed, can eventually also yield equations of Shimura curves.

Remark 5.4. Here is a more concrete way to describe what is going on in Representation B. Let $\{s_0, \dots, s_L\}$ be a basis for V ; then each vector of values $p'_i = [s_0(p_i) : \dots : s_L(p_i)] \in \mathbf{P}^L$ gives the image of the point $p_i \in X$ under the projective embedding. (For convenience, suppose in this remark that the points p_i are all distinct, and that the field F is perfect; it is easy to modify the argument for the general case). We obtain sufficiently many points to be able to identify X as the unique projective curve that interpolates the $\{p'_i\}$, in the sense that X is defined by all the quadric equations vanishing at the $\{p'_i\}$. The quadrics that generate the ideal of X are of the form $\sum_{j,k} c_{jk} X_j X_k$, and can be found by solving for the c_{ij} in the linear system $\{\sum_{j,k} c_{jk} s_j(p_i) s_k(p_i) = 0 \mid 1 \leq i \leq N\}$. Now the individual p_i need not be defined over F , even though the divisor D is F -rational; still, the set of points $\{p'_i\}$ is stable under $\text{Gal}(\bar{F}/F)$, and so the linear system of equations for the c_{jk} is unaffected by the Galois group. This implies that X can be defined by quadrics with F -rational coefficients; for example, take an echelon basis for the solution space of the linear system.

We are now ready for the last result of this article.

Theorem 5.5. *Let $\ell \geq 3$. Fix a number field $F \subset \mathbf{C}$ and an elliptic curve E_0 over F given by a Weierstrass equation $y^2 = x^3 + a_0x + b_0$, with $a_0, b_0 \in F - \{0\}$. Then consider all torsion points $\{(x_{0,P}, y_{0,P}) \mid P \in E_0[\ell](\bar{F}) - \{P_0\}\}$, and the slopes $\lambda_{0,(P)+(Q)+(\ominus P \ominus Q)} = (y_{0,P} - y_{0,Q})/(x_{0,P} - x_{0,Q}) \in F(E_0[\ell])$ of lines through pairs of torsion points (with the appropriate modification when $P = Q$). These slopes for the one elliptic curve E_0 contain enough information to reconstruct the projective embedding of $X(\ell)$ coming from the linear system $\mathcal{M}_2(\Gamma(\ell))$. This embedding is defined over $F(\mu_\ell)$.*

Proof. We first observe that the condition $a_0, b_0 \neq 0$ implies that E_0 does not have nontrivial automorphisms, and hence does not correspond to an elliptic point for $\Gamma(1)$ in the upper half plane \mathcal{H} . Thus the projection map $\pi : X(\ell) \rightarrow X(1)$ is unramified over the point $q_0 \in X(1)(F)$ corresponding to E_0 , and hence the preimages $\{p_1, \dots, p_N\} = \pi^{-1}(\{q_0\})$ are distinct points of $X(\ell)$, which are rational over $F(E_0[\ell])$. We claim that N (which is incidentally $|PSL(2, \mathbf{Z}/\ell\mathbf{Z})|$) is large enough that we can identify modular forms of weight < 12 via their “values” at

the $\{p_i\}$. To see this claim, either use standard formulas for the degree of the line bundle $\mathcal{L}^{\otimes j}$, whose global sections are $\mathcal{M}_j(\Gamma(\ell))$, or note that one section of the line bundle $\mathcal{L}^{\otimes 12}$ is the $\Gamma(1)$ -invariant modular form $b_0^2 a(\tau)^3 - a_0^3 b(\tau)^2$, which vanishes precisely to order 1 at each point p_i ; this last statement holds because modular forms in $\mathcal{M}_{12}(\Gamma(1))$ have precisely one zero (counted appropriately) in the fundamental domain for the $\Gamma(1)$ -action on \mathcal{H} . Thus $N = 12 \deg \mathcal{L}$, and our claim is proved.

Hence, as mentioned in our discussion preceding the theorem, we can represent $X(\ell)$ in Representation B using the line bundle $\hat{\mathcal{L}} = \mathcal{L}^{\otimes 2}$ and the divisor $D = \sum_i (p_i)$; this amounts to representing the spaces V and V' by the “values” of modular forms of weights 2 and 4 at the points $\{p_i\}$. Concretely, such a point p_i corresponds to a choice of symplectic basis $\{T_0, U_0\}$ for the ℓ -torsion $E_0[\ell]$, with $e_\ell(T_0, U_0) = e^{2\pi i/\ell} \in F(E_0[\ell])$. We know how to “evaluate” an Eisenstein series of weight 1 at p_i ; this amounts to computing slopes between the torsion points to get the λ_0 s appearing in the statement of the theorem. Here, the local trivialization of each line bundle $\mathcal{L}^{\otimes j}$ near p_i corresponds to the particular choice of Weierstrass model of E_0 and of its global differential ω_0 . To define this trivialization more precisely, let $\tau_1 \in \mathcal{H}$ be such that the elliptic curve $E_1 = E_{\tau_1} = \mathbf{C}/L_{\tau_1}$ and its symplectic ℓ -torsion basis $\{P_{1/\ell}, P_{\tau_1/\ell}\}$ are isomorphic to our given triple (E_0, T_0, U_0) . Then there exists a unique $u \in \mathbf{C}^\times$ such that $a_0 = u^4 a(\tau_1)$ and $b_0 = u^6 b(\tau_1)$, and which is also compatible with the level structures. Hence each λ_0 is equal to $u \lambda_1(\tau_1)$ for a corresponding classical modular form $\lambda_1(\tau) \in \mathcal{E}_1^\ell$, and similarly for other weights j . It follows that our trivialization of $\mathcal{L}^{\otimes j}$ near p_i is u^j times the trivialization induced by evaluating modular forms in a neighborhood of τ_1 .

At this point, we see that if we work over the field $F_\ell = F(E_0[\ell])$, which contains the values of all the λ_0 s, then our algebra \mathcal{A} is isomorphic to the direct product F_ℓ^N , and our space V (respectively, V') can be obtained as the span of all products of the values of two (respectively, four) of the λ_0 s at each p_i . This follows from Theorems 5.1 and 3.13. We thus obtain equations for $X(\ell)$ from the kernel of $\bar{\mu} : \text{Sym}^2 V \rightarrow V'$. These equations are actually defined over the smaller cyclotomic extension $F(\mu_\ell)$, because our whole setup is invariant under the full $SL(2, \mathbf{Z}/\ell\mathbf{Z})$ -action on modular forms, which permutes the possible symplectic bases for $E_0[\ell]$. Now the action of $\text{Gal}(F_\ell/F(\mu_\ell))$ arises from a subgroup H of $SL(2, \mathbf{Z}/\ell\mathbf{Z})$ (in fact, when E_0 does not have complex multiplication, then for almost all ℓ , $H = SL(2, \mathbf{Z}/\ell\mathbf{Z})$), so the equations that we obtain can be set up over the smaller field $F(\mu_\ell)$, as mentioned in Remark 5.4. \square

We note in closing that an analog of Theorem 5.5 holds for the projective embedding of $X(\ell)$ coming from the (usually incomplete) linear system $\mathcal{E}_1^\ell \subset \mathcal{M}_1(\Gamma(\ell))$. By Theorem 5.1 and a computation of Castelnuovo-Mumford regularity, that projective model is defined by equations in degrees 2 and 3.

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